Logic Exercises

Technical University of Munich
Chair for Logic and Verification

Prof. Tobias Nipkow
Kevin Kappelmann

SS 2021 | Exercise Sheet 11 | 25.06.2021

The tutorials on 02.07 and 09.07 have to be moved. Please participate in the poll on Zulip until 28.06 to find a new date for the tutorials.

Exercise 11.1. [Decidability of Complete Theories]
Assume S is finitely axiomatizable and complete, i.e. $F \in S$ or $\neg F \in S$ for any sentence F.

1. Given only the axiomatization of S, give a procedure deciding whether $S \models F$ for any sentence F.
2. Can you obtain a similar result when the assumption is that the axiom system is only *recursively enumerable*?

Exercise 11.2. [Consequence]
Show that Cn is a closure operator, i.e. Cn fulfills the following properties:

- $S \subseteq Cn(S)$
- if $S \subseteq S'$ then $Cn(S) \subseteq Cn(S')$
- $Cn(Cn(S)) = Cn(S)$

Exercise 11.3. [One Finite, All Finite]
Show that if a theory is finitely axiomatizable, any countable axiomatization of it has a finite subset that axiomatizes the same theory. In other words, if $Cn(\Gamma) = Cn(\Delta)$ with Γ countable and Δ finite, then there is a finite $\Gamma' \subseteq \Gamma$ with $Cn(\Gamma') = Cn(\Gamma)$. Can you also obtain Γ' effectively?

Exercise 11.4. [Natural Deduction]
Prove the following formula using natural deduction.

$$\neg(\forall x(\exists y(\neg P(x) \land P(y))))$$

Homework 11.1. [Counterexamples from Sequent Calculus] (++)
Consider the statement $\forall x(P(x) \to \neg P(f(x)))$.

1. What happens when trying to prove the validity of this formula in sequent calculus?
2. How can we derive a countermodel from the proof tree?
3. Is there a smaller countermodel?

Homework 11.2. [Natural Deduction] (++)
Prove the following statements using natural deduction.

1. $\neg\forall x \exists y \forall z(\neg P(x, z) \land P(z, y))$
2. $\exists x(P(x) \to \forall x P(x))$

Homework 11.3. [Elementary Classes] (++)
In this exercise, we assume that all structures and formulas share the same signature Σ.

We define the operator $Mod(S)$ that returns the class of all structures that model a set of formulas S. In other words, $Mod(S)$ contains all $\mathcal{A}$ such that $\mathcal{A} \models S$.

A class of models M is said to be *elementary* if there is a set of formulas S such that $M = Mod(S)$. If S is just a singleton set, i.e. there is a formula F such that $S = \{F\}$, then M is *basic elementary.*

Prove:

1. A class of models M is basic elementary if and only if there is a *finite* set of formulas S such that $M = Mod(S)$.
2. If M is basic elementary and $M = Mod(S)$ for countable S, then there is a finite subset $S' \subseteq S$ such that $M = Mod(S')$.

> The logic of the world is prior to all truth and falsehood.
>
> — Ludwig Wittgenstein[1]

[1] Yes, Ludwig strikes again – he just dropped too many great quotes.