First-Order Logic
Herbrand Theory
Herbrand universe

The Herbrand universe $T(F)$ of a closed formula $F$ in Skolem form is the set of all terms that can be constructed using the function symbols in $F$.

In the special case that $F$ contains no constants, we first pick an arbitrary constant, say $a$, and then construct the terms.

Formally, $T(F)$ is inductively defined as follows:

- All constants occurring in $F$ belong to $T(F)$; if no constant occurs in $F$, then $a \in T(F)$ where $a$ is some arbitrary constant.

- For every $n$-ary function symbol $f$ occurring in $F$, if $t_1, t_2, \ldots, t_n \in T(F)$ then $f(t_1, t_2, \ldots, t_n) \in T(F)$.

**Note:** All terms in $T(F)$ are variable-free by construction!

**Example**

$F = \forall x \forall y \; P(f(x), g(c, y))$
Herbrand structure

Let $F$ be a closed formula in Skolem form. A structure $\mathcal{A}$ suitable for $F$ is a **Herbrand structure** for $F$ if it satisfies the following conditions:

- $U_{\mathcal{A}} = T(F)$, and
- for every $n$-ary function symbol $f$ occurring in $F$ and every $t_1, \ldots, t_n \in T(F)$: $f^A(t_1, \ldots, t_n) = f(t_1, \ldots, t_n)$.

**Fact**

*If $\mathcal{A}$ is a Herbrand structure, then $\mathcal{A}(t) = t$ for all $t \in U_\mathcal{A}$.***

We call a Herbrand structure that is a model a **Herbrand model**.
Definition

The matrix of a formula $F$ is the result of removing all quantifiers (all $\forall x$ and $\exists x$) from $F$. The matrix is denoted by $F^*$. 
Fundamental theorem of predicate logic

Theorem

Let $F$ be a closed formula in Skolem form. Then $F$ is satisfiable iff it has a Herbrand model.

Proof If $F$ has a Herbrand model then it is satisfiable.

For the other direction let $\mathcal{A}$ be an arbitrary model of $F$. We define a Herbrand structure $\mathcal{T}$ as follows:

- Universe $U_{\mathcal{T}} = T(F)$
- Function symbols $f^\mathcal{T}(t_1, \ldots, t_n) = f(t_1, \ldots, t_n)$
  - If $F$ contains no constant: $a^\mathcal{A} = u$ for some arbitrary $u \in U_{\mathcal{A}}$
- Predicate symbols $(t_1, \ldots, t_n) \in P^\mathcal{T}$ iff $(\mathcal{A}(t_1), \ldots, \mathcal{A}(t_n)) \in P^\mathcal{A}$

Claim: $\mathcal{T}$ is also a model of $F$. 
Claim: $T$ is also a model of $F$.

We prove a stronger assertion:

*For every closed formula $G$ in Skolem form such $G$ contains the same fun. and pred. symbols as $F$: if $A \models G$ then $T \models G$*

**Proof** By induction on the number $n$ of universal quantifiers of $G$.

Basis $n = 0$. Then $G$ has no quantifiers at all. Therefore $A(G) = T(G)$ (why?), and we are done.
Induction step: $G = \forall x \ H$.

$$A \models G$$

$$\Rightarrow \text{ for every } u \in U_A: A[u/x](H) = 1$$

$$\Rightarrow \text{ for every } u \in U_A \text{ of the form } u = A(t)$$

$$\quad \text{where } t \in T(F): A[u/x](H) = 1$$

$$\Rightarrow \text{ for every } t \in T(F): A[A(t)/x](H) = 1$$

$$\Rightarrow \text{ for every } t \in T(F): A(H[t/x]) = 1$$

$$\Rightarrow \text{ for every } t \in T(F): T(H[t/x]) = 1$$

$$\Rightarrow T(t/x)(H) = 1$$

$$\Rightarrow \forall x \ H = 1$$

$$\Rightarrow T \models G$$
Theorem

Let $F$ be a closed formula in Skolem form. Then $F$ is satisfiable iff it has a Herbrand model.

What goes wrong if $F$ is not closed or not in Skolem form?
Herbrand expansion

Let $F = \forall y_1 \ldots \forall y_n F^*$ be a closed formula in Skolem form.
The **Herbrand expansion** of $F$ is the set of formulas

$$E(F) = \{F^*[t_1/y_1] \ldots [t_n/y_n] | t_1, \ldots, t_n \in T(F)\}$$

Informally: the formulas of $E(F)$ are the result of substituting terms from $T(F)$ for the variables of $F^*$ in every possible way.

**Example**

$E(\forall x \forall y P(f(x), g(c, y))) =$

**Note** The Herbrand expansion can be viewed as a set of propositional formulas.
Gödel-Herbrand-Skolem Theorem

Theorem

Let $F$ be a closed formula in Skolem form. Then $F$ is satisfiable iff its Herbrand expansion $E(F)$ is satisfiable (in the sense of propositional logic).

Proof

By the fundamental theorem, it suffices to show: $F$ has a Herbrand model iff $E(F)$ is satisfiable.

Let $F = \forall y_1 \ldots \forall y_n F^*$. $\mathcal{A}$ is a Herbrand model of $F$

iff for all $t_1, \ldots, t_n \in T(F)$, $\mathcal{A}[t_1/y_1] \ldots [t_n/y_n](F^*) = 1$

iff for all $t_1, \ldots, t_n \in T(F)$, $\mathcal{A}(F^*[t_1/y_1] \ldots [t_n/y_n]) = 1$

iff for all $G \in E(F)$, $\mathcal{A}(G) = 1$

iff $\mathcal{A}$ is a model of $E(F)$
Herbrand’s Theorem

Theorem

Let $F$ be a closed formula in Skolem form.
$F$ is unsatisfiable iff some finite subset of $E(F)$ is unsatisfiable.

Proof follows immediately from the Gödel-Herbrand-Skolem Theorem and the Compactness Theorem.
Gilmore’s Algorithm

Let $F$ be a closed formula in Skolem form and let $F_1, F_2, F_3, \ldots$ be an computable enumeration of $E(F)$.

Input: $F$

$n := 0$;

repeat $n := n + 1$;

until $(F_1 \land F_2 \land \ldots \land F_n)$ is unsatisfiable;

return “unsatisfiable”

The algorithm terminates iff $F$ is unsatisfiable.
Semi-decidability Theorems

Theorem
(a) The unsatisfiability problem of predicate logic is (only) semi-decidable.
(b) The validity problem of predicate logic is (only) semi-decidable.

Proof
(a) Gilmore’s algorithm is a semi-decision procedure. (The problem is undecidable. Proof later)
(b) $F$ valid iff $\neg F$ unsatisfiable.
Löwenheim-Skolem Theorem

**Theorem**

*Every satisfiable formula of first-order predicate logic has a model with a countable universe.*

**Proof** Let $F$ be a formula, and let $G$ be an equisatisfiable formula in Skolem form (as produced by the Normal Form transformations). Fact: Every model of $G$ is a model of $F$. (Check this!)

$F$ satisfiable $\implies G$ satisfiable  
$\implies G$ has a Herbrand model $\mathcal{T}$  
$\implies F$ also has that model $\mathcal{T}$  
$\implies F$ has a countable model  
(Herbrand universes are countable)
Löwenheim-Skolem Theorem

Formulas of first-order logic cannot enforce uncountable models

Formulas of first-order logic cannot axiomatize the real numbers because there will always be countable models