Hilbert Systems
Propositional Logic

(See the book by Troelstra and Schwichtenberg)
Easy to define, hard to use
No context management

A Hilber system for propositional logic consists of
  ▶ a set of axioms (formulae)
  ▶ and a single inference rule, $\rightarrow E$ or modus ponens:

$$
\begin{array}{c}
F \rightarrow G \\
\hline
G
\end{array}
\quad
\frac{F}{\rightarrow E}
$$

Proof trees for some Hilbert system are labeled with formulas. The only inference rule is $\rightarrow E$.

Definition
We write $\Gamma \vdash_{\mathcal{H}} F$ if there is a proof tree with root $F$ whose leaves are either axioms or elements of $\Gamma$. 
Alternative proof presentation

Proofs in Hilbert systems are frequently shown as lists of lines

1. $F_1 \quad justification_1$
2. $F_2 \quad justification_2$
   ...
   i. $F_i \quad justification_i$
   ...

$justification_i$ is either assumption, axiom or $\rightarrow E(j, k)$ where $j, k < i$

Like linearized tree but also allows sharing of subproofs
Notational convention:

\[ F \rightarrow G \rightarrow H \text{ means } F \rightarrow (G \rightarrow H) \]

Note:

\[ F \rightarrow G \rightarrow H \equiv F \land G \rightarrow H \]
\[ F \rightarrow G \rightarrow H \not\equiv (F \rightarrow G) \rightarrow H \]
Example (A simple Hilbert system)

Axioms:  

\[ F \rightarrow (G \rightarrow F) \quad (A1) \]

\[ (F \rightarrow G \rightarrow H) \rightarrow (F \rightarrow G) \rightarrow F \rightarrow H \quad (A2) \]

A proof of \( F \rightarrow F \):

\[
\begin{align*}
\rightarrow (F \rightarrow F) & \rightarrow E \\
F \rightarrow F & \rightarrow E \\
\Rightarrow \vdash_H F \rightarrow F
\end{align*}
\]
Theorem (Deduction Theorem)

In any Hilbert-system that contains the axioms A1 and A2:

\[ F, \Gamma \vdash_H G \iff \Gamma \vdash_H F \rightarrow G \]

Proof "\(\iff\)"
\[
\begin{align*}
\Gamma \vdash_H F \rightarrow G & \\
\therefore F, \Gamma \vdash_H F \rightarrow G & \\
\therefore F, \Gamma \vdash_H G & \text{by } \rightarrow E \text{ because } F, \Gamma \vdash_H F
\end{align*}
\]
Theorem (Deduction Theorem)

In any Hilbert-system that contains the axioms A1 and A2:

\[ F, \Gamma \vdash H \iff \Gamma \vdash F \to G \]

Proof “⇒”:
By induction on (the length/depth of) the proof of \( F, \Gamma \vdash H \)
Then by cases on the last proof step:

Case \( G = F \): see proof of \( F \to F \) from A1 and A2
Case \( G \in \Gamma \) or axiom: by A1 and . . .

Case \( \to E \) from \( H \to G \) and \( H \):

\[
(F \to H \to G) \to (F \to H) \to F \to G \quad F \to H \to G
\]

\[
(F \to H) \to F \to G
\]

\[
F \to H
\]

\[
F \to G
\]
Hilbert System

From now on $\vdash_{H}$ refers to the following set of axioms:

\begin{align*}
F \rightarrow G & \rightarrow F & (A1) \\
(F \rightarrow G \rightarrow H) & \rightarrow (F \rightarrow G) \rightarrow F \rightarrow H & (A2) \\
F \rightarrow G & \rightarrow F \land G & (A3) \\
F \land G & \rightarrow F & (A4) \\
F \land G & \rightarrow G & (A5) \\
F & \rightarrow F \lor G & (A6) \\
G & \rightarrow F \lor G & (A7) \\
F \lor G & \rightarrow (F \rightarrow H) \rightarrow (G \rightarrow H) \rightarrow H & (A8) \\
(\neg F \rightarrow \bot) & \rightarrow F & (A9)
\end{align*}
Relating
Hilbert and Natural Deduction
Theorem (Hilbert can simulate ND)

If $\Gamma \vdash_N F$ then $\Gamma \vdash_H F$

Proof translation in two steps: $\vdash_N \rightsquigarrow \vdash_H + \to I \rightsquigarrow \vdash_H$

1. Transform a ND-proof tree into a proof tree containing Hilbert axioms, $\to E$ and $\to I$
   by replacing all other ND rules by Hilbert proofs incl. $\to I$
   Principle: ND rule $\rightsquigarrow$ 1 axiom + $\to I/E$

2. Eliminate the $\to I$ rules by the Deduction Theorem
Lemma (ND can simulate Hilbert)

\[ \text{If } \Gamma \vdash_H F \text{ then } \Gamma \vdash_N F \]

\textbf{Proof} by induction on \( \Gamma \vdash_H F \).

- Every Hilbert axiom is provable in ND (Exercise!)
- \( \rightarrow \text{E} \) is also available in ND

\textbf{Corollary}

\[ \Gamma \vdash_H F \iff \Gamma \vdash_N F \]

\textbf{Corollary (Soundness and completeness)}

\[ \Gamma \vdash_H F \iff \Gamma \models F \]