Propositional Logic
Horn Formulas

Efficient satisfiability checks

In the following:

- A very efficient satisfiability check for the special class of Horn formulas.
- Efficient satisfiability checks for arbitrary formulas in CNF: resolution (later).

Horn formulas

Definition

A formula F in CNF is a Horn formula if every disjunction in F contains at most one positive literal.

A disjunction in a Horn formula can equivalently be viewed as an implication $K \rightarrow B$ where K is a conjunction of atoms or $K = \top$ and B is an atom or $B = \bot$:

$$\begin{aligned} (\neg A \vee \neg B \vee C) &\equiv (A \wedge B \rightarrow C) \\ (\neg A \vee \neg B) &\equiv (A \wedge B \rightarrow \bot) \\ A &\equiv (\top \rightarrow A) \end{aligned}$$

Satisfiablity check for Horn formulas

Input: a Horn formula F.

Algorithm building a model (assignment) $\mathcal{M}$:

for all atoms A_i in F **do** $\mathcal{M}(A_i) := 0$;
while F has a subformula $K \to B$
such that $\mathcal{M}(K) = 1$ and $\mathcal{M}(B) = 0$
do
if $B = \perp$ **then return** "unsatisfiable"
else $\mathcal{M}(B) := 1$

return "satisfiable"

Maximal number of iterations of the while loop:
number of implications in F

Each iteration requires at most $O(|F|)$ steps.

Overall complexity: $O(|F|^2)$

[Algorithm can be improved to $O(|F|)$. See Schöning.]

Correctness of the model building algorithm

Theorem

The algorithm returns "satisfiable" iff F is satisfiable.

Proof Observe: if the algorithm sets $\mathcal{M}(B) = 1$, then $\mathcal{A}(B) = 1$ for every assignment $\mathcal{A}$ such that $\mathcal{A}(F) = 1$. This is an invariant.

(a) If "unsatisfiable" then unsatisfiable.

We prove unsatisfiability by contradiction.

Assume $\mathcal{A}(F) = 1$ for some $\mathcal{A}$.

Let $(A_{i_1} \wedge \ldots \wedge A_{i_k} \to \bot)$ be the subformula causing "unsatisfiable".

Since $\mathcal{M}(A_{i_1}) = \cdots = \mathcal{M}(A_{i_k}) = 1$, $\mathcal{A}(A_{i_1}) = \ldots = \mathcal{A}(A_{i_k}) = 1$.

Then $\mathcal{A}(A_{i_1} \wedge \ldots \wedge A_{i_k} \to \bot) = 0$ and so $\mathcal{A}(F) = 0$, contradiction.

So F has no satisfying assignments.

(b) If "satisfiable" then satisfiable.

After termination with "satisfiable",
for every subformula $K \to B$ of F, $\mathcal{M}(K) = 0$ or $\mathcal{M}(B) = 1$.
Therefore $\mathcal{M}(K \to B) = 1$ and thus $\mathcal{M} \models F$.
In fact, the invariant shows that $\mathcal{M}$ is the minimal model of F.