Natural Deduction
Propositional Logic

(See the book by Troelstra and Schwichtenberg)

Natural deduction (Gentzen 1935) aims at *natural* proofs

It formalizes good mathematical practice

Resolution but also sequent calculus aim at proof search

Main principles

1. For every logical operator $\oplus$ there are two kinds of rules:

Introduction rules: How to prove $F \oplus G$

$$\frac{\ldots}{F \oplus G}$$

Elimination rules What can be proved from $F \oplus G$

$$\frac{F \oplus G \quad \ldots}{\ldots}$$

Examples

$$\frac{A \quad B}{A \wedge B}\ \wedge I \qquad \frac{F \wedge G}{F}\ \wedge E_1 \qquad \frac{F \wedge G}{G}\ \wedge E_2$$

Main principles

2. Proof can contain subproofs with *local/closed* assumptions

Example

If from the local assumption F we can prove G
then we can prove $F \rightarrow G$.

The formal inference rule:

$$\dfrac{\begin{array}{c}[F] \\ \vdots \\ G\end{array}}{F \rightarrow G} \rightarrow I$$

A proof tree:

$$\dfrac{\dfrac{[P] \quad Q}{P \wedge Q} \wedge I}{P \rightarrow P \wedge Q} \rightarrow I$$

Form the (open) assumption Q we can prove $P \rightarrow P \wedge Q$.
In symbols: $Q \vdash_N P \rightarrow P \wedge Q$

Growing the proof tree

Upwards:

$$\dfrac{\dfrac{[P] \quad Q}{P \wedge Q}\ \wedge I}{P \rightarrow P \wedge Q}\ {\rightarrow}I$$

Downwards:

$$\dfrac{\dfrac{[P] \quad Q}{P \wedge Q}\ \wedge I}{P \rightarrow P \wedge Q}\ {\rightarrow}I$$

ND proof trees

The nodes of a ND proof tree are labeled by formulas.
Leaf nodes represent assumptions.
The root node is the conclusion.
Assumptions can be open or closed.
Closed assumptions are written [F].

Intuition:

- Open assumptions are used in the proof of the conclusion
- Closed assumptions are local assumptions in a subproof that have been closed (removed) by some proof rule like $\rightarrow I$.

ND proof trees are defined inductively.

- Every F is a ND proof tree (with open assumption F and conclusion F). Reading: From F we can prove F.
- New proof trees are constructed by the rules of ND.

Natural Deduction rules

$$\frac{F \quad G}{F \wedge G}\ \wedge I \qquad\qquad \frac{F \wedge G}{F}\ \wedge E_1 \quad \frac{F \wedge G}{G}\ \wedge E_2$$

$$\frac{\begin{array}{c}[F]\\ \vdots \\ G\end{array}}{F \to G}\ {\to} I \qquad\qquad \frac{F \to G \quad F}{G}\ {\to} E$$

$$\frac{F}{F \vee G}\ \vee I_1 \quad \frac{G}{F \vee G}\ \vee I_2 \qquad \frac{F \vee G \quad \begin{array}{c}[F]\\ \vdots \\ H\end{array} \quad \begin{array}{c}[G]\\ \vdots \\ H\end{array}}{H}\ \vee E$$

$$\frac{\begin{array}{c}[\neg F]\\ \vdots \\ \bot\end{array}}{F}\ \bot$$

Natural Deduction rules

Rules for $\neg$ are special cases of rules for $\rightarrow$:

$$\dfrac{\begin{matrix}[F] \\ \vdots \\ \bot\end{matrix}}{\neg F}\ \neg I \qquad \dfrac{\neg F \quad F}{\bot}\ \neg E$$

Natural Deduction rules

How to read a rule

$$\dfrac{\dots \quad \begin{matrix}[F] \\ \vdots \\ G\end{matrix} \quad \dots}{\dots}\; r$$

Forward:
Close all (or some) of the assumptions F in the proof of G when applying rule r

Backward:
In the subproof of G you can use the local assumption $[F]$.

Can use labels to show which rule application closed which assumptions.

Soundness

Definition

$\Gamma \vdash_N F$ if there is a proof tree with root F and open assumptions contained in the set of formulas Γ.

Lemma (Soundness)

If $\Gamma \vdash_N F$ *then* $\Gamma \models F$

Proof by induction on the depth of the proof tree for $\Gamma \vdash_N F$.

Base case: no rule, $F \in \Gamma$

Step: Case analysis of last rule

Case $\rightarrow E$:
IH: $\Gamma \models F \rightarrow G \qquad \Gamma \models F$
To show: $\Gamma \models G$
Assume $\mathcal{A} \models \Gamma \Rightarrow^{IH} \mathcal{A}(F \rightarrow G) = 1$ and $\mathcal{A}(F) = 1 \Rightarrow \mathcal{A}(G) = 1$

Soundness

Case

$$\dfrac{\begin{array}{c}[F]\\ \vdots\\ G\end{array}}{F \to G} \to I$$

IH: $\Gamma, F \models G$
To show: $\Gamma \models F \to G$
iff for all $\mathcal{A}$, $\mathcal{A} \models \Gamma \Rightarrow \mathcal{A} \models F \to G$
iff for all $\mathcal{A}$, $\mathcal{A} \models \Gamma \Rightarrow (\mathcal{A} \models F \Rightarrow \mathcal{A} \models G)$
iff for all $\mathcal{A}$, $\mathcal{A} \models \Gamma$ and $\mathcal{A} \models F \;\Rightarrow\; \mathcal{A} \models G$
iff IH

Completeness

Towards completeness

ND can simulate truth tables

Lemma (Tertium non datur)

$\vdash_N F \vee \neg F$

Corollary (Cases)

If $F, \Gamma \vdash_N G$ *and* $\neg F, \Gamma \vdash_N G$ *then* $\Gamma \vdash_N G$.

Definition

$$F^{\mathcal{A}} := \begin{cases} F & \text{if } \mathcal{A}(F) = 1 \\ \neg F & \text{if } \mathcal{A}(F) = 0 \end{cases}$$

Towards completeness

Lemma (1)

If $atoms(F) \subseteq \{A_1, \ldots, A_n\}$ *then* $A_1^{\mathcal{A}}, \ldots, A_n^{\mathcal{A}} \vdash_N F^{\mathcal{A}}$

Proof by induction on F

Lemma (2)

If $atoms(F) = \{A_1, \ldots, A_n\}$ *and* $\models F$
then $A_1^{\mathcal{A}}, \ldots, A_k^{\mathcal{A}} \vdash_N F$ *for all* $k \leq n$

Proof by (downward) induction on $k = n, \ldots, 0$

Completeness

Theorem (Completeness)

If $\Gamma \models F$ *then* $\Gamma \vdash_N F$

Proof

Relating
Sequent Calculs and Natural Deduction

Constructive approach to relating proof systems:

- Show how to transform proofs in one system into proofs in another system
- Implicit in inductive (meta)proof

Theorem (ND can simulate SC)

If $\vdash_G \Gamma \Rightarrow \Delta$ *then* $\Gamma, \neg\Delta \vdash_N \bot$ *(where* $\neg\{F_1, \dots\} = \{\neg F_1, \dots\}$*)*

Proof by induction on (the depth of) $\vdash_G \Gamma \Rightarrow \Delta$

Corollary (Completeness of ND)

If $\Gamma \models F$ *then* $\Gamma \vdash_N F$

Proof If $\Gamma \models F$ then $\Gamma_0 \models F$ for some finite $\Gamma_0 \subseteq \Gamma$.

Two completness proofs

- Direct
- By simulating a complete system

Theorem (SC can simulate ND)

If $\Gamma \vdash_N F$ *and* Γ *is finite then* $\vdash_G \Gamma \Rightarrow F$

Proof by induction on $\Gamma \vdash_N F$

Corollary

If $\Gamma \vdash_N F$ *then there is some finite* $\Gamma_0 \subseteq \Gamma$ *such that* $\vdash_G \Gamma_0 \Rightarrow F$