First-Order Logic

Basic Proof Theory

Gebundene Namen sind Schall und Rauch

We permit ourselves to identifty formulas that differ only in the names of bound variables.

Example

$\forall x \exists y\, P(x, y) \ = \ \forall u \exists v\, P(u, v)$

The renaming must not capture free variables:

$\forall x\, P(x, y) \ \neq \ \forall y\, P(y, y)$

Substitution $F[t/x]$ assumes that bound variables in F are automatically renamed to avoid capturing free variables in t.

Example

$(\forall x\, P(x, y))[f(x)/y] \ = \forall x'\, P(x', f(x))$

All proof systems below are extensions
of the corresponding propositional systems

Sequent Calculus

Sequent Calculus rules

$$\frac{F[t/x], \forall x\, F, \Gamma \Rightarrow \Delta}{\forall x\, F, \Gamma \Rightarrow \Delta}\ \forall L \qquad \frac{\Gamma \Rightarrow F[y/x], \Delta}{\Gamma \Rightarrow \forall x\, F, \Delta}\ \forall R(*)$$

$$\frac{F[y/x], \Gamma \Rightarrow \Delta}{\exists x\, F, \Gamma \Rightarrow \Delta}\ \exists L(*) \qquad \frac{\Gamma \Rightarrow F[t/x], \exists x\, F, \Delta}{\Gamma \Rightarrow \exists x\, F, \Delta}\ \exists R$$

$(*)$: y not free in the conclusion of the rule

Note: $\forall L$ and $\exists R$ do not delete the principal formula

Soundness

Lemma

For every quantifier rule $\dfrac{S'}{S}$, $|S|$ *and* $|S'|$ *are equivalid.*

Theorem (Soundness)

If $\vdash_G S$ *then* $\models |S|$.

Proof induction on the size of the proof of $\vdash_G S$
using the above lemma and the corresponding propositional lemma
$(|S| \equiv |S_1| \wedge \ldots \wedge |S_n|)$.

Completeness Proof

Construct counter model
from (possibly infinite!) failed proof search

Let $e_0, e_1, \ldots$ be an enumeration of all terms
(over some given set of function symbols and variables)

Proof search

Construct proof tree incrementally:

1. Pick some uproved leaf $\Gamma \Rightarrow \Delta$ such that some rule is applicable.
2. Pick some principal formula in $\Gamma \Rightarrow \Delta$ fairly and apply rule.
   $\forall R$, $\exists L$: pick some arbitrary new y
   $\forall L$, $\exists R$:
   $$t = \begin{cases} e_0 & \text{if the p.f. has never been instantiated} \\ & \text{(on the path to the root)} \\ e_{i+1} & \text{if the previous instantiation of the p.f.} \\ & \text{(on the path to the root) used } e_i \end{cases}$$

Failed proof search: there is a branch A such that A ends in a sequent where no rule is applicable or A is infinite.

Construction of Herbrand countermodel $\mathcal{A}$ from A

$$
\begin{aligned}
U_{\mathcal{A}} &= \text{all terms over the function symbols and variables in } A \\
f^{\mathcal{A}}(t_1, \ldots, t_n) &= f(t_1, \ldots, t_n) \\
P^{\mathcal{A}} &= \{(t_1, \ldots, t_n) \mid P(t_1, \ldots, t_n) \in \Gamma \text{ for some } \Gamma \Rightarrow \Delta \in A\}
\end{aligned}
$$

Theorem

For all $\Gamma \Rightarrow \Delta \in A$*:* $\mathcal{A}(F) = \begin{cases} 1 & \text{if } F \in \Gamma \\ 0 & \text{if } F \in \Delta \end{cases}$

Proof by induction on the structure of F

$F = P(t_1, \ldots, t_n)$:

$F \in \Gamma \Rightarrow \mathcal{A}(F) = 1$ by def

$F \in \Delta \Rightarrow F \notin$ any $\Gamma \in A$, (A would end in Ax) $\Rightarrow \mathcal{A}(F) = 0$

F not atomic $\Rightarrow F$ must be p.f. in some $\Gamma \Rightarrow \Delta \in A$ (fairness!)

Let $\Gamma' \Rightarrow \Delta'$ be the next sequent in A

$F = \neg G$: $F \in \Gamma$ iff $G \in \Delta'$ iff $\mathcal{A}(G) = 0$ (IH) iff $\mathcal{A}(F) = 1$

$F = G_1 \wedge G_2$:

$F \in \Gamma \Rightarrow G_1, G_2 \in \Gamma' \Rightarrow A(G_1) = \mathcal{A}(G_2) = 1$ (IH) $\Rightarrow \mathcal{A}(F) = 1$

$F \in \Delta \Rightarrow G_1 \in \Delta'$ or $G_2 \in \Delta' \Rightarrow \mathcal{A}(G_1) = 0$ or $\mathcal{A}(G_2) = 0$ (IH)

$\Rightarrow \mathcal{A}(F) = 0$

$F = \forall x\, G$: $F \in \Delta \Rightarrow G[y/x] \in \Delta' \Rightarrow \mathcal{A}(G[y/x]) = 0$ (IH)

$\Rightarrow \mathcal{A}[\mathcal{A}(y)/x](G) = 0 \Rightarrow \mathcal{A}(F) = 0$

Completeness

Corollary

If proof search with root $\Gamma \Rightarrow \Delta$ *fails,*
then there is a structure $\mathcal{A}$ *such that* $\mathcal{A}(\bigwedge \Gamma \to \bigvee \Delta) = 0$*.*

Example

$\exists x\, P(x) \Rightarrow \forall x\, P(x)$

Corollary (Completeness)

If $\models |\Gamma \to \Delta|$ *then* $\vdash_G \Gamma \Rightarrow \Delta$

Proof by contradiction. If not $\vdash_G \Gamma \Rightarrow \Delta$ then proof search fails.
Then there is an $\mathcal{A}$ such that $\mathcal{A}(\bigwedge \Gamma \to \bigvee \Delta) = 0$.
Therefore not $\models |\Gamma \to \Delta|$.

Natural Deduction

Natural Deduction rules

$$\frac{F[y/x]}{\forall x\, F}\ \forall I(*) \qquad \frac{\forall x\, F}{F[t/x]}\ \forall E$$

$$\frac{F[t/x]}{\exists x\, F}\ \exists I \qquad \frac{\exists x\, F \qquad \begin{array}{c}[F[y/x]]\\ \vdots \\ H\end{array}}{H}\ \exists E(**)$$

$(*)$: $(y = x$ or $y \notin fv(F))$ and
y not free in an open assumption in the proof of $F[y/x]$

$(**)$: $(y = x$ or $y \notin fv(F))$ and
y not free in H or in an open assumption in the proof of the second premise, except for $F[y/x]$

Theorem (Soundness)

If $\Gamma \vdash_N F$ *then* $\Gamma \models F$

Proof as before, with additional cases:

$$\dfrac{\exists x\, F \qquad \begin{matrix}[F[y/x]] \\ \vdots \\ H\end{matrix}}{H}\ \exists E(**)$$

IH: $\Gamma \models \exists x F$ and $F[y/x], \Gamma \models H$

Show $\Gamma \models H$. Assume $\mathcal{A} \models \Gamma$.

$\Rightarrow \mathcal{A} \models \exists x\, F$ (by IH) $\Rightarrow$ there is a $u \in U_{\mathcal{A}}$ s.t. $\mathcal{A}[u/x] \models F$

$\Rightarrow \mathcal{A}[u/y] \models F[y/x]$ because $y = x$ or $y \notin fv(F)$

$\mathcal{A}[u/y] \models \Gamma$ because y not free in Γ

$\Rightarrow \mathcal{A}[u/y] \models H$ by IH

$\Rightarrow \mathcal{A} \models H$ because y not free in proof of 2nd prem.

Theorem (ND can simulate SC)

If $\vdash_G \Gamma \Rightarrow \Delta$ *then* $\Gamma, \neg\Delta \vdash_N \bot$ *(where* $\neg\{F_1, \dots\} = \{\neg F_1, \dots\}$*)*

Proof by induction on (the depth of) $\vdash_G \Gamma \Rightarrow \Delta$

Corollary (Completeness of ND)

If $\Gamma \models F$ *then* $\Gamma \vdash_N F$

Proof as before: compactness, completeness of $\vdash_G$, translation to $\vdash_N$

Translation from $\vdash_N$ to $\vdash_G$ also as before: $I \mapsto R$, $E \mapsto L + cut$

Equality

Hilbert System

Hilbert System

Additional rule $\forall I$:
if F is provable then $\forall y\, F[y/x]$ is provable
provided x not free in the assumptions and ($y = x$ or $y \notin fv(F)$)

Additional axioms:
$\forall x\, F \to F[t/x]$
$F[t/x] \to \exists x\, F$
$\forall x(G \to F) \to (G \to \forall y\, F[y/x]) \quad (*)$
$\forall x(F \to G) \to (\exists y\, F[y/x] \to G) \quad (*)$

$(*)$ if $x \notin fv(G)$ and ($y = x$ or $y \notin fv(F)$)

Equivalence of Hilbert and ND

As before, with additional cases.