Sequent Calculus
Propositional Logic

Sequent Calculus

Invented by Gerhard Gentzen in 1935. Birth of proof theory.

Proof rules

$$\frac{S_1 \quad \dots \quad S_n}{S}$$

where $S_1, \dots S_n$ and S are sequents

$$\Gamma \Rightarrow \Delta$$

where Γ and Δ are finite multisets of formulas.
(Multiset = set with possibly repeated elements)
(Could use sets instead of multisets
but this causes some complications)

Important: $\Rightarrow$ is just a separator
Formally, a sequent is a pair of finite multisets.

Intuition: $\Gamma \Rightarrow \Delta$ is provable iff $\bigwedge \Gamma \to \bigvee \Delta$ is a tautology

Sequents: Notation

- We use set notation for multisets, eg $\{A, B \to C, A\}$
- Drop $\{\}$: $F_1, \ldots, F_m \Rightarrow G_1, \ldots G_n$
- F, Γ abbreviates $\{F\} \cup \Gamma$ (similarly for Δ)
- Γ_1, Γ_2 abbreviates $\Gamma_1 \cup \Gamma_2$ (similarly for Δ)

Sequent Calculus rules

Intuition: read backwards as proof search rules

$$\frac{}{\bot, \Gamma \Rightarrow \Delta}\ \bot L \qquad \frac{}{A, \Gamma \Rightarrow A, \Delta}\ Ax$$

$$\frac{\Gamma \Rightarrow F, \Delta}{\neg F, \Gamma \Rightarrow \Delta}\ \neg L \qquad \frac{F, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \neg F, \Delta}\ \neg R$$

$$\frac{F, G, \Gamma \Rightarrow \Delta}{F \wedge G, \Gamma \Rightarrow \Delta}\ \wedge L \qquad \frac{\Gamma \Rightarrow F, \Delta \quad \Gamma \Rightarrow G, \Delta}{\Gamma \Rightarrow F \wedge G, \Delta}\ \wedge R$$

$$\frac{F, \Gamma \Rightarrow \Delta \quad G, \Gamma \Rightarrow \Delta}{F \vee G, \Gamma \Rightarrow \Delta}\ \vee L \qquad \frac{\Gamma \Rightarrow F, G, \Delta}{\Gamma \Rightarrow F \vee G, \Delta}\ \vee R$$

$$\frac{\Gamma \Rightarrow F, \Delta \quad G, \Gamma \Rightarrow \Delta}{F \rightarrow G, \Gamma \Rightarrow \Delta}\ {\rightarrow} L \qquad \frac{F, \Gamma \Rightarrow G, \Delta}{\Gamma \Rightarrow F \rightarrow G, \Delta}\ {\rightarrow} R$$

Every rule decomposes its principal formula

Example

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P, Q \lor \neg R \Rightarrow P, Q}\,Ax \qquad \dfrac{\dfrac{}{R, Q \Rightarrow P, Q}\,Ax \qquad \dfrac{\dfrac{}{R \Rightarrow R, P, Q}\,Ax}{R, \neg R \Rightarrow P, Q}\,\neg L}{R, Q \lor \neg R \Rightarrow P, Q}\,\lor L}{P \lor R, Q \lor \neg R \Rightarrow P, Q}\,\lor L}{P \lor R, Q \lor \neg R \Rightarrow P \lor Q}\,\lor R}{(P \lor R) \land (Q \lor \neg R) \Rightarrow P \lor Q}\,\land L}{\Rightarrow (P \lor R) \land (Q \lor \neg R) \to P \lor Q}\,\to R
$$

$$
\dfrac{F, \Gamma \Rightarrow G, \Delta}{\Gamma \Rightarrow F \to G, \Delta}\,\to R \quad \dfrac{F, G, \Gamma \Rightarrow \Delta}{F \land G, \Gamma \Rightarrow \Delta}\,\land L \quad \dfrac{\Gamma \Rightarrow F, G, \Delta}{\Gamma \Rightarrow F \lor G, \Delta}
$$

Proof search properties

- For every logical operator ($\neg$ etc)
  there is one left and one right rule
- Every formula in the premise of a rule
  is a subformula of the conclusion of the rule.
  This is called the subformula property.
  $\Rightarrow$ no need to guess anything when applying a rule backward
- Backward rule application terminates
  because one operator is removed in each step.

Instances of rules

Definition

An instance of a rule is the result of
replacing Γ and Δ by multisets of concrete formulas
and F and G by concrete formulas.

Example

$$\frac{\Rightarrow P \wedge Q, A, B}{\neg(P \wedge Q) \Rightarrow A, B}$$

is an instance of

$$\frac{\Gamma \Rightarrow F, \Delta}{\neg F, \Gamma \Rightarrow \Delta}$$

setting $F := P \wedge Q$, $\Gamma := \emptyset$, $\Delta := \{A, B\}$

Proof trees

Definition (Proof tree)

A proof tree is a tree whose nodes are sequents and where each parent-children fragment

$$\frac{S_1 \quad \dots \quad S_n}{S}$$

is an instance of a proof rule.

($\Rightarrow$ all leaves must be instances of axioms)

A sequent S is provable if there is a proof tree with root S.
Then we write $\vdash_G S$.

Proof trees

An alternative inductive definition of proof trees:

Definition (Proof tree)

If

$$\frac{S_1 \quad \dots \quad S_n}{S}$$

is an instance of a proof rule
and there are proof trees $T_1, \dots T_n$ with roots $S_1, \dots, S_n$ then

$$\frac{T_1 \quad \dots \quad T_n}{S}$$

is a proof tree (with root S).

What does $\Gamma \Rightarrow \Delta$ "mean"?

Definition

$$|\Gamma \Rightarrow \Delta| \;=\; (\bigwedge \Gamma \to \bigvee \Delta)$$

Example: $|\{A, B\} \Rightarrow \{P, Q\}| = (A \wedge B \to P \vee Q)$

Remember: $\bigwedge \emptyset = \top$ and $\bigvee \emptyset = \bot$

Aim: $\vdash_G S$ iff $|S|$ is a tautology

Lemma (Rule Equivalence)

For every rule $\dfrac{S_1 \quad \ldots \quad S_n}{S}$

- $|S| \equiv |S_1| \wedge \ldots \wedge |S_n|$
- $|S|$ *is a tautology iff all* S_i *are tautologies*

Theorem (Soundness of $\vdash_G$)

If $\vdash_G S$ *then* $\models |S|$.

Proof by induction on the height of the proof tree for $\vdash_G S$.
Tree must end in rule instance

$$\frac{S_1 \quad \dots \quad S_n}{S}$$

IH: $\models S_i$ for all i.
Thus $\models |S|$ by the previous lemma.

Proof Search and Completeness

Proof search = growing a proof tree from the root

- Start from an initial sequent S_0
- At each stage we have some potentially *partial* proof tree with unproved leaves
- In each step, pick some unproved leaf S and some rule instance

$$\frac{S_1 \quad \dots \quad S_n}{S}$$

  and extend the tree with that rule instance
  (creating new unproved leaves $S_1, \dots, S_n$)

Proof search termintes if ...

- there are no more unproved leaves — success
- there is some unproved leaf where no rule applies — failure
  $\Rightarrow$ that leaf is of the form

$$P_1, \ldots, P_k \Rightarrow Q_1, \ldots, Q_l$$

where all P_i and Q_j are atoms, no $P_i = Q_j$ and no $P_i = \bot$

Example (failed proof)

$$\dfrac{\dfrac{\dfrac{}{P \Rightarrow P}\,Ax \quad Q \Rightarrow P}{P \vee Q \Rightarrow P}\,\vee L \quad \dfrac{P \Rightarrow Q \quad \dfrac{}{Q \Rightarrow Q}\,Ax}{P \vee Q \Rightarrow Q}\,\vee L}{P \vee Q \Rightarrow P \wedge Q}\,\wedge R$$

Falsifying assignments?

Proof search = Counterexample search

Can view sequent calculus as a search for a falsifying assignment for $|\Gamma \Rightarrow \Delta|$:

Make Γ true and Δ false

Some examples:

$$\frac{F, G, \Gamma \Rightarrow \Delta}{F \wedge G, \Gamma \Rightarrow \Delta} \wedge L$$

To make $F \wedge G$ true, make both F and G true

$$\frac{\Gamma \Rightarrow F, \Delta \quad \Gamma \Rightarrow G, \Delta}{\Gamma \Rightarrow F \wedge G, \Delta} \wedge R$$

To make $F \wedge G$ false, make F or G false

Lemma (Search Equivalence)

At each stage of the search process,
if $S_1, \ldots, S_k$ are the unproved leaves, then $|S_0| \equiv |S_1| \wedge \ldots \wedge |S_k|$

Proof by induction on the number of search steps.
Initially trivially true (base case).
When applying a rule instance

$$\frac{U_1 \quad \ldots \quad U_n}{S_i}$$

we have

$$\begin{aligned} |S_0| &\equiv |S_1| \wedge \ldots \wedge |S_i| \wedge \ldots \wedge |S_k| && \text{(by IH)} \\ &\equiv |S_1| \wedge \cdots \wedge |S_{i-1}| \wedge |U_1| \wedge \cdots \wedge |U_n| \wedge |S_{i+1}| \wedge \ldots \wedge |S_k| \end{aligned}$$

by Lemma Rule Equivalence.

Lemma

If proof search fails, $|S_0|$ is not a tautology.

Proof If proof search fails, there is some unproved leaf $S =$

$$P_1, \dots, P_k \Rightarrow Q_1, \dots, Q_l$$

where no $P_i = Q_j$ and no $P_i = \bot$.
This sequent can be falsified by setting $\mathcal{A}(P_i) := 1$ (for all i) and $\mathcal{A}(Q_j) := 0$ (for all j) and all other atoms to 0 or 1.
Thus $\mathcal{A}(|S|) = 0$ and hence $\mathcal{A}(S_0) = 0$ by Lemma Search Equivalence. $\square$

Because of soundness of $\vdash_G$:

Corollary

Starting with some fixed S_0, proof search cannot both fail (for some choices) and succeed (for other choices).

$\Rightarrow$ no need for backtracking upon failure!

Lemma

Proof search terminates.

Proof In every step, one logical operator is removed.
$\Rightarrow$ size of sequent decreases by 1
$\Rightarrow$ Depth of proof tree is bounded by size of S_0
but breadth only bounded by $2^{\text{size of } S_0}$

Corollary

Proof search is a decision procedure: it either succeeds or fails.

Theorem (Completeness)

If $\models |S|$ *then* $\vdash_G S$.

Proof by contraposition: if not $\vdash_G S$ then proof seach must fail.
Therefore $\not\models |S|$.

Multisets versus sets

Termination only because of multisets.
With sets, the principal formula may get duplicated:

$$\frac{\Gamma \Rightarrow F, \Delta}{\neg F, \Gamma \Rightarrow \Delta}\,\neg L \quad \overset{\Gamma := \{\neg F\}}{\rightsquigarrow} \quad \frac{\neg F \Rightarrow F, \Delta}{\neg F \Rightarrow \Delta}$$

An alternative formulation of the set version:

$$\frac{\Gamma \backslash \{\neg F\} \Rightarrow F, \Delta}{\neg F, \Gamma \Rightarrow \Delta}$$

Gentzen used sequences (hence "sequent calculus")

Admissible Rules and Cut Elimination

Admissible rules

Definition

A rule

$$\frac{S_1 \quad \dots \quad S_n}{S}$$

is admissible if $\vdash_G S_1, \dots, \vdash_G S_n$ together imply $\vdash_G S$.

$\Rightarrow$ Admissible rules can be used in proofs like normal rules

Admissibility is often proved by induction.

Aim: prove admissibility of

$$\frac{\Gamma \Rightarrow F, \Delta \quad \Gamma, F \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ cut$$

This is Gentzen's *Hauptsatz*. Many applications.

Lemma (Non-atomic Ax)

The non-atomic axiom rule

$$\dfrac{}{F, \Gamma \Rightarrow F, \Delta}\ Ax'$$

is admissible, i.e. $\vdash_G F, \Gamma \Rightarrow F, \Delta$.

Proof idea: decompose F, then use Ax.
Formally: proof by induction on (the structure of) F.
Case $F_1 \to F_2$:

$$\dfrac{\dfrac{\dfrac{}{F_1, \Gamma \Rightarrow F_1, F_2, \Delta}\ IH \qquad \dfrac{}{F_1, F_2, \Gamma \Rightarrow F_2, \Delta}\ IH}{F_1, F_1 \to F_2 \Rightarrow F_2, \Delta}\ {\to}L}{F_1 \to F_2, \Gamma \Rightarrow F_1 \to F_2, \Delta}\ {\to}R$$

The other cases are analogous.

Semantic proofs of admissibility

Admissibility of

$$\frac{S_1 \quad \ldots \quad S_n}{S}$$

can also be shown semantically (using $\vdash_G = \models$)
by proving that $\models |S_1|, \ldots, \models |S_n|$ together imply $\models |S|$.

Semantic proofs are *much simpler*
and much less informative than syntactic proofs.
Syntactic proofs show *how* to eliminate admissible rules.
For examle, the admissibility proof of Ax' is a recursive procedure that decomposes F. In particular it tells us that the elimination of Ax' generates a proof of size $O(\qquad)$.

We focuses on proof theory

Weakening

Notation:
$\Gamma \Rightarrow_n \Delta$ means that there is a proof tree for $\Gamma \Rightarrow \Delta$ of depth $\leq n$.

Lemma (Weakening)

If $\Gamma \Rightarrow_n \Delta$ *then* $\Gamma', \Gamma \Rightarrow_n \Delta', \Delta$.

Proof idea: take proof tree for $\Gamma \Rightarrow \Delta$
and add Γ' everywhere on the left and Δ' everywhere on the right.

General principal: transform proof trees

Notation:
$D : \Gamma \Rightarrow \Delta$ means that D is a proof tree for $\Gamma \Rightarrow \Delta$

Inversion rules

Lemma (Inversion rules)

$\wedge L^{-1}$ *If* $F \wedge G, \Gamma \Rightarrow_n \Delta$ *then* $F, G, \Gamma \Rightarrow_n \Delta$

$\vee R^{-1}$ *If* $\Gamma \Rightarrow_n F \vee G, \Delta$ *then* $\Gamma \Rightarrow_n F, G, \Delta$

$\wedge R^{-1}$ *If* $\Gamma \Rightarrow_n F_1 \wedge F_2, \Delta$ *then* $\Gamma \Rightarrow_n F_i, \Delta$ $(i = 1, 2)$

$\vee L^{-1}$ *If* $F_1 \vee F2, \Gamma \Rightarrow_n \Delta$ *then* $F_i, \Gamma \Rightarrow_n \Delta$ $(i = 1, 2)$

$\rightarrow R^{-1}$ *If* $\Gamma \Rightarrow_n F \rightarrow G, \Delta$ *then* $F, \Gamma \Rightarrow_n G, \Delta$

$\rightarrow L^{-1}$ *If* $F \rightarrow G, \Gamma \Rightarrow_n \Delta$ *then* $\Gamma \Rightarrow_n F, \Delta$ *and* $G, \Gamma \Rightarrow_n \Delta$

$$\frac{F, G, \Gamma \Rightarrow \Delta}{F \wedge G, \Gamma \Rightarrow \Delta} \wedge L \quad \frac{\Gamma \Rightarrow F, G, \Delta}{\Gamma \Rightarrow F \vee G, \Delta} \vee R \quad \frac{\Gamma \Rightarrow F, \Delta \quad \Gamma \Rightarrow}{\Gamma \Rightarrow F \wedge G,}$$

Negation?

Proof of $\to L^{-1}$

If $F \to G, \Gamma \Rightarrow_n \Delta$ then $\Gamma \Rightarrow_n F, \Delta$ and $G, \Gamma \Rightarrow_n \Delta$

Proof by induction on n. Base case trivial because $\Rightarrow_0$ impossible.
Assume $D : F \to G, \Gamma \Rightarrow_{n+1} \Delta$
Let r be the last rule in D. Proof by cases.

Case $r = Ax$ ($r = \bot L$ similar)

$\Rightarrow \; D = \dfrac{}{F \to G, A, \Gamma' \Rightarrow_1 A, \Delta'}$ where $\Gamma = A, \Gamma'$ and $\Delta = A, \Delta'$

$\Rightarrow \; \overline{\Gamma \Rightarrow_1 F, \Delta}$ and $\overline{G, \Gamma \Rightarrow_1 \Delta}$

Otherwise there are two subcases.

1. $F \to G$ is the principal formula

$$\Rightarrow \; D = \dfrac{\Gamma \Rightarrow_{n+1} F, \Delta \quad G, \Gamma \Rightarrow_n \Delta}{F \to G, \Gamma \Rightarrow_n \Delta} \to L$$

Proof of $\to L^{-1}$

If $F \to G, \Gamma \Rightarrow_n \Delta$ then $\Gamma \Rightarrow_n F, \Delta$ and $G, \Gamma \Rightarrow_n \Delta$

2. $F \to G$ is not the principal formula
Cases r:
Case $r = \vee R$

$$D = \frac{F \to G, \Gamma \Rightarrow_{n+1} H_1, H_2, \Delta'}{F \to G, \Gamma \Rightarrow_n H_1 \vee H_2, \Delta'}$$

IH: $\dfrac{\Gamma \Rightarrow_n F, H_1, H_2, \Delta'}{\Gamma \Rightarrow_{n+1} F, \Delta} \vee R$ and $\dfrac{G, \Gamma \Rightarrow_n H_1, H_2, \Delta'}{G, \Gamma \Rightarrow_{n+1} \Delta} \vee R$

Similar for all other rules because $F \to G$ is not principal

Contraction

$$\frac{F, F, \Gamma \Rightarrow \Delta}{F, \Gamma \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow F, F, \Delta}{\Gamma \Rightarrow F, \Delta}$$

Lemma (Contraction)

(i) *If* $F, F, \Gamma \Rightarrow_n \Delta$ *then* $F, \Gamma \Rightarrow_n \Delta$

(ii) *If* $\Gamma \Rightarrow_n F, F, \Delta$ *then* $\Gamma \Rightarrow_n F, \Delta$

Proof by induction on n. Base case trivial. Step: focus on (i).
Assume $D: \ F, F, \Gamma \Rightarrow_{n+1} \Delta$
Let r be the last rule in D. Proof by cases.

Case $r = \rightarrow L$ (other rules similar)
Two subcases:
1. F is not principal formula

$$\Rightarrow D = \frac{F, F, \Gamma' \Rightarrow_n G, \Delta \quad F, F, H, \Gamma' \Rightarrow_n \Delta}{F, F, G \rightarrow H, \Gamma' \Rightarrow_{n+1} \Delta} \rightarrow L$$

$$\text{IH: } \frac{F, \Gamma' \Rightarrow_n G, \Delta \quad F, H, \Gamma' \Rightarrow_n \Delta}{F, G \rightarrow H, \Gamma' \Rightarrow \Delta} \rightarrow L$$

Contraction

2. F is principal formula

$$\Rightarrow D = \frac{G \to H, \Gamma \Rightarrow_n G, \Delta \quad H, G \to H, \Gamma \Rightarrow_n \Delta}{G \to H, G \to H, \Gamma \Rightarrow_{n+1} \Delta} \to L$$

No $\perp R$

Lemma

If $\vdash_G \Gamma \Rightarrow \Delta$ *then* $\vdash_G \Gamma \Rightarrow \Delta - \{\perp\}$

Proof idea:

- no rule expects $\perp$ on the right
- no rule can move $\perp$ from right to left.

$\Rightarrow$ no rule is disabled by removing $\perp$ on the right
$\Rightarrow$ the same proof rules that prove $\Gamma \Rightarrow \Delta$ also prove $\Gamma \Rightarrow \Delta - \{\perp\}$.
Formally: induction on the height of the proof tree for $\Gamma \Rightarrow \Delta$
$=$ recursive transformation of proof tree.

Atomic cut

Lemma (Atomic cut)

If $D_1 : \Gamma \Rightarrow A, \Delta$ *and* $D_2 : A, \Gamma \Rightarrow \Delta$ *then* $\vdash_G \Gamma \Rightarrow \Delta$

Proof by induction on the depth of D_1.

Cut

Theorem (Cut)

If $D_1 : \Gamma \Rightarrow F, \Delta$ *and* $D_2 : F, \Gamma \Rightarrow \Delta$ *then* $\vdash_G \Gamma \Rightarrow \Delta$

Proof by induction on F.