Tableaux Calculus
Propositional Logic

A compact version of sequent calculus

The idea

What's "wrong" with sequent calculus:

Why do we have to copy(?) Γ and Δ
with every rule application?

The answer: tableaux calculus.
The idea:

Describe *backward* sequent calculus rule application
but leave Γ and Δ implicit/shared

Comparison:

Sequent	Proof is a tree labeled by sequents, trees grow upwards
Tableaux	Proof is a tree labeled by formulas, trees grow downwards

Terminology: tableau = tableaux calculus proof tree

Tableaux rules (examples)

Notation: $+F \;\approx\; F$ occurs on the right of $\Rightarrow$
$\; -F \;\approx\; F$ occurs on the left of $\Rightarrow$

$$\textit{S.C.} \qquad\qquad \textit{Tab.} \qquad\qquad \textit{Effect}$$

$$\frac{F, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \neg F, \Delta} \quad\rightsquigarrow\quad \frac{+\neg F}{-F} \quad\rightsquigarrow\quad \begin{array}{c} +\neg F \\ | \\ -F \end{array}$$

$$\frac{\Gamma \Rightarrow F, G, \Delta}{\Gamma \Rightarrow F \vee G, \Delta} \quad\rightsquigarrow\quad \frac{+F \vee G}{\begin{array}{c} +F \\ +G \end{array}} \quad\rightsquigarrow\quad \begin{array}{c} +F \vee G \\ | \\ +F \\ | \\ +G \end{array}$$

$$\frac{\Gamma \Rightarrow F, \Delta \quad \Gamma \Rightarrow G, \Delta}{\Gamma \Rightarrow F \wedge G, \Delta} \quad\rightsquigarrow\quad \frac{+F \wedge G}{+F \mid +G} \quad\rightsquigarrow\quad \begin{array}{c} +F \wedge G \\ / \quad \backslash \\ +F \qquad +G \end{array}$$

Interpretation of tableaux rule

$$\frac{F}{FGH}$$

if F matches the formula at some node in the tableau extend the end of some branch starting at that node according to FGH.

Example

$$
\begin{array}{c}
-\ A \to B \\
-\ B \to C \\
-\ A \\
+\ C
\end{array}
$$

$A \to B, B \to C, A \Rightarrow C$

From tableau to sequents:

- Every path from the root to a leaf in a tableau represents a sequent
- The set of all such sequents represents the set of leaves of the corresponding sequent calculus proof

$\Rightarrow$

- A branch is closed (proved) if both $+F$ and $-F$ occur on it or $-\bot$ occurs on it
- The root sequent is proved if all branches are closed

Algorithm to prove $F_1, \ldots \Rightarrow G_1, \ldots$:

1. Start with the tableau $-F_1, \ldots, +G_1, \ldots$.
2. while there is an open branch do
   pick some non-atomic formula on that branch,
   extend the branch according to the matching rule

Termination

No formula needs to be used twice on the same branch.
But possibly on *different* branches:

$$+\neg A \land \neg B$$
$$+A \lor B$$

A formula occurrence in a tableau can be deleted
if it has been used in every unclosed branch
starting from that occurrence

Tableaux rules

$$\frac{-\neg F}{+F} \qquad \frac{+\neg F}{-F}$$

$$\frac{-F \wedge G}{\begin{array}{c}-F\\-G\end{array}} \qquad \frac{+F \wedge G}{+F \mid +G}$$

$$\frac{-F \vee G}{-F \mid -G} \qquad \frac{+F \vee G}{\begin{array}{c}+F\\+G\end{array}}$$

$$\frac{-F \rightarrow G}{+F \mid -G} \qquad \frac{+F \rightarrow G}{\begin{array}{c}-F\\+G\end{array}}$$