First-order Predicate Logic

Theories
Definitions

Definition
A signature \( \Sigma \) is a set of predicate and function symbols.
A \( \Sigma \)-formula is a formula that contains only predicate and function symbols from \( \Sigma \).
A \( \Sigma \)-structure is a structure that interprets all predicate and function symbols from \( \Sigma \).

Definition
A sentence is a closed formula.
In the sequel, \( S \) is a set of sentences.
Theories

Definition
A theory is a set of sentences $S$ such that $S$ is closed under consequence: If $S \models F$ and $F$ is closed, then $F \in S$.

Let $\mathcal{A}$ be a $\Sigma$-structure:
$Th(\mathcal{A})$ is the set of all sentences true in $\mathcal{A}$:
$Th(\mathcal{A}) = \{F \mid F$ $\Sigma$-sentence and $\mathcal{A} \models F\}$

Lemma
Let $\mathcal{A}$ be a $\Sigma$-structure and $F$ a $\Sigma$-sentence.
Then $\mathcal{A} \models F$ iff $Th(\mathcal{A}) \models F$.

Corollary
$Th(\mathcal{A})$ is a theory.
Lemma

Let $\mathcal{A}$ be a $\Sigma$-structure and $F$ a $\Sigma$-sentence.

Then $\mathcal{A} \models F$ iff $Th(\mathcal{A}) \models F$.

Proof

“$\Rightarrow$”

Assume $\mathcal{A} \models F$

To show $Th(\mathcal{A}) \models F$, assume $\mathcal{B} \models Th(\mathcal{A})$ and show $\mathcal{B} \models F$

$\Rightarrow$ for all $G \in Th(\mathcal{A})$, $\mathcal{B} \models G$

$\Rightarrow \mathcal{B} \models F$ because $F \in Th(\mathcal{A})$

“$\Leftarrow$”:

Assume $Th(\mathcal{A}) \models F$

$\Rightarrow$ for all $\mathcal{B}$, if $\mathcal{B} \models Th(\mathcal{A})$ then $\mathcal{B} \models F$

$\Rightarrow \mathcal{A} \models F$ because $\mathcal{A} \models Th(\mathcal{A})$
**Example**

**Notation:** \((\mathbb{Z}, +, \leq)\) denotes the structure with universe \(\mathbb{Z}\) and the standard interpretations for the symbols + and \(\leq\). The same notation is used for other standard structures where the interpretation of a symbol is clear from the symbol.

**Example (Linear integer arithmetic)**

\(Th(\mathbb{Z}, +, \leq)\) is the set of all sentences over the signature \(+, \leq\) that are true in the structure \((\mathbb{Z}, +, \leq)\).
Famous numerical theories

\[ Th(\mathbb{R}, +, \leq) \] is called linear real arithmetic.
It is decidable.

\[ Th(\mathbb{R}, +, \ast, \leq) \] is called real arithmetic.
It is decidable.

\[ Th(\mathbb{Z}, +, \leq) \] is called linear integer arithmetic or Presburger arithmetic.
It is decidable.

\[ Th(\mathbb{Z}, +, \ast, \leq) \] is called integer arithmetic.
It is not even semidecidable (= r.e.).

Decidability via special algorithms.
Consequences

Definition
Let $S$ be a set of $\Sigma$-sentences.

$Cn(S)$ is the set of consequences of $S$:
$Cn(S) = \{ F \mid F \text{ $\Sigma$-sentence and } S \models F \}$

Examples

$Cn(\emptyset)$ is the set of valid sentences.

$Cn(\{\forall x \forall y \forall z \ (x \ast y) \ast z = x \ast (y \ast z)\})$ is the set of sentences that are true in all semigroups.

Lemma

If $S$ is a set of $\Sigma$-sentences, $Cn(S)$ is a theory.

Proof
Assume $F$ is closed and $Cn(S) \models F$. Show $F \in Cn(S)$, i.e. $S \models F$. Assume $\mathcal{A} \models S$. Thus $\mathcal{A} \models Cn(S)$ (*) and hence $\mathcal{A} \models F$, i.e. $S \models F$. (*) Assume $G \in Cn(S)$, i.e. $S \models G$. With $\mathcal{A} \models S$ the desired $\mathcal{A} \models G$ follows.
Axioms

Definition
Let $S$ be a set of $\Sigma$-sentences.

A theory $T$ is **axiomatized** by $S$ if $T = \text{Cn}(S)$

A theory $T$ is **axiomatizable** if there is some decidable or recursively enumerable $S$ that axiomatizes $T$.

A theory $T$ is **finitely axiomatizable** if there is some finite $S$ that axiomatizes $T$. 
Completeness and elementary equivalence

**Definition**
A theory $T$ is **complete** if for every sentence $F$, $T \models F$ or $T \models \neg F$.

**Fact**
Th($A$) is complete.

**Example**
$Cn(\{\forall x \forall y \forall z (x \ast y) \ast z = x \ast (y \ast z)\})$ is incomplete:
neither $\forall x \forall y x \ast y = y \ast x$ nor its negation are present.

**Definition**
Two structures $A$ and $B$ are **elementarily equivalent** if $Th(A) = Th(B)$.

**Theorem**
A theory $T$ is complete iff all its models are elementarily equivalent.
Theorem

A theory $T$ is complete iff all its models are elementarily equivalent.

Proof If $T$ is unsatisfiable, then $T$ is complete (because $T \models F$ for all $F$) and all models are elementarily equivalent.

Now assume $T$ has a model $\mathcal{M}$.

$\rightarrow$

Assume $T$ is complete. Let $F \in Th(\mathcal{M})$.

We cannot have $T \models \neg F$ because $\mathcal{M} \models T$ would imply $\mathcal{M} \models \neg F$

but $\mathcal{M} \models F$ because $F \in Th(\mathcal{M})$. Thus $T \models F$ by completeness.

Therefore every formula that is true in some model of $T$ is true in all models of $T$.

$\leftarrow$

Assume all models of $T$ are elem.eq. Let $F$ be closed.

Either $\mathcal{M} \models F$ or $\mathcal{M} \models \neg F$. By elem.eq. $T \models F$ or $T \models \neg F$.

Why? Assume $\mathcal{M} \models F$ (similar for $\mathcal{M} \models \neg F$).

To show $T \models F$, assume $A \models T$ and show $A \models F$.

$\Rightarrow Th(A) = Th(\mathcal{M})$ by elem.eq.

$\Rightarrow$ for all closed $F$, $A \models F$ iff $\mathcal{M} \models F$

$\Rightarrow A \models F$ because $\mathcal{M} \models F$