Logic Exercises

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Exercise 3.1. [**System G1c**]
An alternative definition of the sequent calculus ("G1c") is defined as follows, where A, B are formulas:

Axioms

$$\text{Ax}\ \ A \Rightarrow A \qquad\qquad \text{L}\bot\ \ \bot \Rightarrow$$

Rules for weakening (W) and contraction (C)

$$\text{LW}\ \frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \qquad\qquad \text{RW}\ \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}$$

$$\text{LC}\ \frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \qquad\qquad \text{RC}\ \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}$$

Rules for the logical operators

$$\text{L}\wedge\ \frac{A_i, \Gamma \Rightarrow \Delta}{A_0 \wedge A_1, \Gamma \Rightarrow \Delta}\ (i = 0, 1) \qquad \text{R}\wedge\ \frac{\Gamma \Rightarrow \Delta, A \qquad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}$$

$$\text{L}\vee\ \frac{A, \Gamma \Rightarrow \Delta \qquad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta} \qquad \text{R}\vee\ \frac{\Gamma \Rightarrow \Delta, A_i}{\Gamma \Rightarrow \Delta, A_0 \vee A_1}\ (i = 0, 1)$$

$$\text{L}\rightarrow\ \frac{\Gamma \Rightarrow \Delta, A \qquad B, \Gamma \Rightarrow \Delta}{A \rightarrow B, \Gamma \Rightarrow \Delta} \qquad \text{R}\rightarrow\ \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \rightarrow B}$$

Notably, weaking and contraction are built-in rules. Moreover, for system G1c, we define $\neg F := F \rightarrow \bot$.

Show that the sequent calculus from the lecture can be simulated by G1c, that is $\vdash_G \Gamma \Rightarrow \Delta$ implies $\vdash_{G1c} \Gamma \Rightarrow \Delta$.

Exercise 3.2. [**Cut Elimination, Semantically**]
Semantically prove the admissibility of the following rule:

$$\text{If } \vdash_G \Gamma \Rightarrow F, \Delta \text{ and } \vdash_G F, \Gamma \Rightarrow \Delta \text{ then } \vdash_G \Gamma \Rightarrow \Delta$$

Exercise 3.3. [**More Connectives**]
Define simple and correct left and right sequent rules for the logical connectives "nand" ($\overline{\wedge}$) and "xor" ($\otimes$).

Exercise 3.4. **[Inversion]**
Show that the following inversion rule is admissible using proof transformations:

$$\frac{\Gamma \Rightarrow F \vee G, \Delta}{\Gamma \Rightarrow F, G, \Delta}$$

Homework 3.1. **[Stay Classy]** (++)

1. Prove the formulas $F \vee \neg F$ (tertium non datur) and $(\neg F \to F) \to F$ (consequentia mirabilis) in System G1c.
2. The intuitionistic system "G1I" is the subsystem of G1c obtained by restricting all rules to sequents with at most one succedent formula and by replacing the rule $L{\to}$ with

$$\frac{\Gamma \Rightarrow F \qquad G, \Gamma \Rightarrow H}{F \to G, \Gamma \Rightarrow H}$$

Fact: neither of former formulas are provable in G1I (try it). Are their doubly negated forms $\neg\neg(F \vee \neg F)$ and $\neg\neg((\neg F \to F) \to F)$ provable?

Homework 3.2. **[Inversion Rules]** (++)

Show that the following inversion rules are admissible using proof transformations:

$$\frac{F \wedge G, \Gamma \Rightarrow \Delta}{F, G, \Gamma \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow F \to G, \Delta}{F, \Gamma \Rightarrow G, \Delta}$$

Homework 3.3. **[What Could Go Wrong?]** (++)

We consider the following "alternatives" for the rules $(\vee L)$ and $(\vee R)$ from the lecture:

$$(\vee L)' \ \frac{F, G, \Gamma \Rightarrow \Delta}{F \vee G, \Gamma \Rightarrow \Delta} \qquad (\vee R)' \ \frac{\Gamma \Rightarrow F, \Delta \qquad \Gamma \Rightarrow G, \Delta}{\Gamma \Rightarrow F \vee G, \Delta}$$

1. Prove that $(\vee L)'$ is unsound.
2. Prove that $(\vee R)'$ is sound.
3. Consider the sequent calculus where $(\vee R)$ is replaced by $(\vee R)'$. Is the calculus sound?
4. Consider the following unsuccessful derivation using the sequent calculus with $(\vee R)'$:

$$(\to L)\ \dfrac{(\neg R)\ \dfrac{(\vee R)'\ \dfrac{(Ax)\ \dfrac{}{X \Rightarrow X} \qquad X \Rightarrow Y}{X \Rightarrow X \vee Y}}{\Rightarrow \neg X, X \vee Y} \qquad \dfrac{Y \Rightarrow X \qquad \dfrac{}{Y \Rightarrow Y}\ (Ax)'}{Y \Rightarrow X \vee Y}\ (\vee R)'}{\neg X \to Y \Rightarrow X \vee Y}$$

Explain: how is it possible that neither $\mathcal{A} := \{X \mapsto 1, Y \mapsto 0\}$ nor $\mathcal{A}' := \{X \mapsto 0, Y \mapsto 1\}$ is a countermodel for the sequent $\neg X \to Y \Rightarrow X \vee Y$? What property is rule $(\vee R)'$ missing that would allow us to conclude that $\mathcal{A}$ and $\mathcal{A}'$ are countermodels?

> If I had a world of my own, everything would be nonsense.
>
> — Alice