Technical University of Munich
Chair for Logic and Verification

Prof. Tobias Nipkow
Kevin Kappelmann

SS 2022 | Exercise Sheet 3 | 13.05.2022

Exercise 3.1. **[System G1c]**
An alternative definition of the sequent calculus ("G1c") is defined as follows, where A, B are formulas:

Axioms

$$\text{Ax}\ A \Rightarrow A \qquad\qquad \text{L}\bot\ \bot \Rightarrow$$

Rules for weakening (W) and contraction (C)

$$\text{LW}\ \frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \qquad\qquad \text{RW}\ \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}$$

$$\text{LC}\ \frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \qquad\qquad \text{RC}\ \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}$$

Rules for the logical operators

$$\text{L}\wedge\ \frac{A_i, \Gamma \Rightarrow \Delta}{A_0 \wedge A_1, \Gamma \Rightarrow \Delta}\ (i = 0, 1) \qquad \text{R}\wedge\ \frac{\Gamma \Rightarrow \Delta, A \qquad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}$$

$$\text{L}\vee\ \frac{A, \Gamma \Rightarrow \Delta \qquad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta} \qquad \text{R}\vee\ \frac{\Gamma \Rightarrow \Delta, A_i}{\Gamma \Rightarrow \Delta, A_0 \vee A_1}\ (i = 0, 1)$$

$$\text{L}\rightarrow\ \frac{\Gamma \Rightarrow \Delta, A \qquad B, \Gamma \Rightarrow \Delta}{A \rightarrow B, \Gamma \Rightarrow \Delta} \qquad \text{R}\rightarrow\ \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \rightarrow B}$$

Notably, weaking and contraction are built-in rules. Moreover, for system G1c, we define $\neg F := F \rightarrow \bot$.

Show that the sequent calculus from the lecture can be simulated by G1c, that is $\vdash_G \Gamma \Rightarrow \Delta$ implies $\vdash_{G1c} \Gamma \Rightarrow \Delta$.

Solution:
The rules Ax and $\bot L$ are simulated by Ax and $L\bot$ together with the weakening rules (by induction on $|\Gamma| + |\Delta|$).

We consider two rules for logical operators: $\wedge L$ and $\neg R$. The other cases are similar. We show how those can be simulated in G1c.

$$\text{LC}\ \cfrac{L\wedge\ \cfrac{L\wedge\ \cfrac{\mathbf{F}, \mathbf{G}, \mathbf{\Gamma} \Rightarrow \mathbf{\Delta}}{F, F \wedge G, \Gamma \Rightarrow \Delta}}{F \wedge G, F \wedge G, \Gamma \Rightarrow \Delta}}{\mathbf{F} \wedge \mathbf{G}, \mathbf{\Gamma} \Rightarrow \mathbf{\Delta}} \qquad \cfrac{R\rightarrow\ \cfrac{\text{RW}\ \cfrac{\mathbf{F}, \mathbf{\Gamma} \Rightarrow \mathbf{\Delta}}{F, \Gamma \Rightarrow \bot, \Delta}}{\Gamma \Rightarrow F \rightarrow \bot, \Delta}}{\mathbf{\Gamma} \Rightarrow \neg\mathbf{F}, \mathbf{\Delta}}$$

Exercise 3.2. **[Cut Elimination, Semantically]**
Semantically prove the admissibility of the following rule:

$$\text{If } \vdash_G \Gamma \Rightarrow F, \Delta \text{ and } \vdash_G F, \Gamma \Rightarrow \Delta \text{ then } \vdash_G \Gamma \Rightarrow \Delta$$

Solution:
To prove this semantically, we have to show that given $|\Gamma \Rightarrow F, \Delta|$ and $|F, \Gamma \Rightarrow \Delta|$, $|\Gamma \Rightarrow \Delta|$ holds. In fact, an even stronger property holds: precedent and antecedent are equivalent.

Let A, C, L be arbitrary and assume $\models A \to (L \vee C)$ (1) and $\models L \wedge A \to C$ (2). We have to show that $\models A \to C$. Pick an assignment $\mathcal{A}$ with $\mathcal{A}(A) = 1$ (3). Then by (1) we have $\mathcal{A}(L \vee C) = 1$. Hence, either $\mathcal{A}(L) = 1$ or $\mathcal{A}(C) = 1$. In the latter case, we are done. In the former case, we obtain $\mathcal{A}(C)$ with (2,3).

Exercise 3.3. **[More Connectives]**
Define simple and correct left and right sequent rules for the logical connectives "nand" ($\bar{\wedge}$) and "xor" ($\otimes$).

Solution:
The simplest way to derive the sequent rules is to consider a definition of $\bar{\wedge}$ and $\otimes$ in terms of other connectives with existing rules:

$$\begin{aligned} F \mathbin{\bar{\wedge}} G &\equiv \neg(F \wedge G) \\ F \otimes G &\equiv (F \wedge \neg G) \vee (\neg F \wedge G) \end{aligned}$$

One can apply the sequent calculus rules on these definitions and simplify accordingly to obtain:

$$\bar{\wedge}L \; \frac{\Gamma \Rightarrow \Delta, F \qquad \Gamma \Rightarrow \Delta, G}{\Gamma, F \mathbin{\bar{\wedge}} G \Rightarrow \Delta} \qquad \bar{\wedge}R \; \frac{\Gamma, F, G \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, F \mathbin{\bar{\wedge}} G}$$

$$\otimes L \; \frac{\Gamma, F \Rightarrow \Delta, G \qquad \Gamma, G \Rightarrow \Delta, F}{\Gamma, F \otimes G \Rightarrow \Delta} \qquad \otimes R \; \frac{\Gamma \Rightarrow \Delta, F, G \qquad \Gamma, F, G \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, F \otimes G}$$

Exercise 3.4. **[Inversion]**
Show that the following inversion rule is admissible using proof transformations:

$$\frac{\Gamma \Rightarrow F \vee G, \Delta}{\Gamma \Rightarrow F, G, \Delta}$$

Solution:
Proof by induction on the depth of the premise's proof tree, denoted by n. The base case is vacuous because there are no proof trees of depth 0. In case $n+1$, proceed by case analysis on the final rule of the premise's proof tree:

The cases $\bot L$ and Ax are trivial (you should ask yourself "why?" and be able to answer it).

Now if $F \vee G$ is the principal formula, then the only applicable rule is $\vee R$. Hence, as the premise of $\vee R$, we obtain a derivation $\Gamma \Rightarrow_n F, G, \Delta$ and we are done.

If $F \vee G$ is not the principal formula, in all cases, we apply the inductive hypotheses to all proof trees of the rule's assumptions and then reapply the rule to the obtained proofs to construct our final derivation (cf lecture).

Here is the case for $\wedge R$: We have $\Delta = F' \wedge G', \Delta'$ and

$$\frac{\Gamma \Rightarrow_n F \vee G, F', \Delta' \qquad \Gamma \Rightarrow_n F \vee G, G', \Delta'}{\Gamma \Rightarrow_{n+1} F \vee G, F' \wedge G', \Delta'} \ (\wedge R)$$

by assumption. Thus, by the IH, we obtain $\Gamma \Rightarrow_n F, G, F', \Delta'$ and $\Gamma \Rightarrow_n F, G, G', \Delta'$. Applying $\wedge R$ to those proofs finishes the proof:

$$\frac{\Gamma \Rightarrow_n F, G, F', \Delta' \qquad \Gamma \Rightarrow_n F, G, G', \Delta'}{\Gamma \Rightarrow_{n+1} F, G, F' \wedge G', \Delta'} \ (\wedge R)$$

Homework 3.1. **[Stay Classy]** (++)

1. Prove the formulas $F \lor \neg F$ (tertium non datur) and $(\neg F \to F) \to F$ (consequentia mirabilis) in System G1c.
2. The intuitionistic system "G1l" is the subsystem of G1c obtained by restricting all rules to sequents with at most one succedent formula and by replacing the rule $L{\to}$ with

$$\dfrac{\Gamma \Rightarrow F \qquad G, \Gamma \Rightarrow H}{F \to G, \Gamma \Rightarrow H}$$

   Fact: neither of former formulas are provable in G1l (try it). Are their doubly negated forms $\neg\neg(F \lor \neg F)$ and $\neg\neg((\neg F \to F) \to F)$ provable?

Solution:

1. The proofs are (almost) syntax-directed. Here's the proof of the former:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{F \Rightarrow F, \bot}(RW + Ax)}{\Rightarrow F, \neg F}(R{\to})}{\Rightarrow F \lor \neg F, F \lor \neg F}(2 * R\lor)}{\Rightarrow F \lor \neg F}(RC)$$

2. Though the formulas are not derivable in G1l, their doubly negated forms indeed are. Here's the proof for the latter formula: Let $G \coloneqq (\neg F \to F) \to F$, then

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\neg G \Rightarrow G}(*) \qquad \dfrac{}{\neg G, \bot \Rightarrow \bot}(LW + L\bot)}{\neg G, \neg G \Rightarrow \bot}(L{\to})}{\neg G \Rightarrow \bot}(LC)}{\Rightarrow \neg\neg G}(R{\to})$$

   where (*) is:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{F, \neg F \to F \Rightarrow F}(LW + Ax)}{F \Rightarrow G}(R{\to}) \qquad \dfrac{}{F, \bot \Rightarrow \bot}(LW + Ax)}{\neg G, F \Rightarrow \bot}(L{\to})}{\neg G \Rightarrow \neg F}(R{\to}) \qquad \dfrac{}{\neg G, F \Rightarrow F}(LW + Ax)}{\neg G, \neg F \to F \Rightarrow F}(L{\to})}{\neg G \Rightarrow G}(R{\to})$$

Homework 3.2. **[Inversion Rules]** (++)
Show that the following inversion rules are admissible using proof transformations:

$$\frac{F \wedge G, \Gamma \Rightarrow \Delta}{F, G, \Gamma \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow F \to G, \Delta}{F, \Gamma \Rightarrow G, \Delta}$$

Solution:
The proofs are by induction on the depth of the premise's proof tree. They are analagous to the inversion lemma proofs done in the lecture and in the tutorial.

Homework 3.3. **[What Could Go Wrong?]** (++)
We consider the following "alternatives" for the rules $(\vee L)$ and $(\vee R)$ from the lecture:

$$(\vee L)' \ \frac{F, G, \Gamma \Rightarrow \Delta}{F \vee G, \Gamma \Rightarrow \Delta} \qquad (\vee R)' \ \frac{\Gamma \Rightarrow F, \Delta \qquad \Gamma \Rightarrow G, \Delta}{\Gamma \Rightarrow F \vee G, \Delta}$$

1. Prove that $(\vee L)'$ is unsound.
2. Prove that $(\vee R)'$ is sound.
3. Consider the sequent calculus where $(\vee R)$ is replaced by $(\vee R)'$. Is the calculus sound?
4. Consider the following unsuccessful derivation using the sequent calculus with $(\vee R)'$:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{X \Rightarrow X}(Ax) \qquad X \Rightarrow Y}{X \Rightarrow X \vee Y}(\vee R)'}{\Rightarrow \neg X, X \vee Y}(\neg R) \qquad \dfrac{Y \Rightarrow X \qquad \dfrac{}{Y \Rightarrow Y}(Ax)'}{Y \Rightarrow X \vee Y}(\vee R)'}{\neg X \to Y \Rightarrow X \vee Y}(\to L)$$

   Explain: how is it possible that neither $\mathcal{A} := \{X \mapsto 1, Y \mapsto 0\}$ nor $\mathcal{A}' := \{X \mapsto 0, Y \mapsto 1\}$ is a countermodel for the sequent $\neg X \to Y \Rightarrow X \vee Y$? What property is rule $(\vee R)'$ missing that would allow us to conclude that $\mathcal{A}$ and $\mathcal{A}'$ are countermodels?

Solution:

1. $A \wedge B \models A \wedge B$ but $A \vee B \not\models A \wedge B$.
2. Assume (1) $\bigwedge A_i \models F \vee \bigvee C_i$ and (2) $\bigwedge A_i \models G \vee \bigvee C_i$. If $\bigwedge A_i \models \bigvee C_i$, then also $\bigwedge A_i \models F \vee G \vee \bigvee C_i$. If $\bigwedge A_i \not\models \bigvee C_i$, then $\bigwedge A_i \models F$ from assumption (1) (note that we do not need assumption (2)). Hence also $\bigwedge A_i \models F \vee G \bigvee C_i$.
3. Yes. We just proved the rule sound and all other rules were proven sound in the lecture. Soundness of the new calculus then follows again by a simple induction on proof trees.
4. Rule $(\vee R)'$ is not an equivalence, that is its inversion is not sound. Hence, we may conclude that the sequent is valid if search is successful but not the other other way round (the calculus is incomplete).

> If I had a world of my own, everything would be nonsense.
>
> — Alice