Technical University of Munich
Chair for Logic and Verification

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Exercise 4.1. **[Natural Deduction]**
Prove the following formulas by natural deduction:

1. $(F \land G) \land H \to F \land (G \land H)$
2. $(F \lor G) \lor H \to F \lor (G \lor H)$
3. $\neg(F \land G) \to (\neg F \lor \neg G)$

Solution:

1)

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}[(F\land G)\land H]^1\\ \vdots\\ (F\land G)\land H\end{matrix}}{F\land G}}{F}\land E \qquad \dfrac{\dfrac{\dfrac{\begin{matrix}[(F\land G)\land H]^1\\ \vdots\end{matrix}}{(F\land G)\land H}}{\dfrac{F\land G}{G}\land E}\land E \qquad \dfrac{\begin{matrix}[(F\land G)\land H]^1\\ \vdots\\ (F\land G)\land H\end{matrix}}{H}\land E}{G\land H}\land I}{F\land(G\land H)}\land I}{(F\land G)\land H \to F\land(G\land H)}\to I\ (1)$$

2)

$$\dfrac{\dfrac{\begin{matrix}[(F\lor G)\lor H]^1\\ \vdots\\ (F\lor G)\lor H\end{matrix} \qquad \dfrac{\begin{matrix}[F\lor G]^2\\ \vdots\\ F\lor G\end{matrix} \qquad \dfrac{\begin{matrix}[F]^4\\ \vdots\\ F\end{matrix}}{F\lor(G\lor H)}\lor I\ ^{(4)} \qquad \dfrac{\dfrac{\begin{matrix}[G]^5\\ \vdots\\ G\end{matrix}}{G\lor H}\lor I}{F\lor(G\lor H)}\lor I\ ^{(5)}}{F\lor(G\lor H)}\lor E\ (2) \qquad \dfrac{\dfrac{\begin{matrix}[H]^3\\ \vdots\\ H\end{matrix}}{G\lor H}\lor I}{F\lor(G\lor H)}\lor I\ (3)}{F\lor(G\lor H)}\lor E}{(F\lor G)\lor H \to F\lor(G\lor H)}\to I\ (1)$$

3)

$$\dfrac{\dfrac{\dfrac{\begin{matrix}[\neg(F\land G)]^1\\ \vdots\\ \neg(F\land G)\end{matrix} \qquad \dfrac{\text{(similar)}\ F \qquad \dfrac{\dfrac{\begin{matrix}[\neg(\neg F\lor\neg G)]^2\\ \vdots\\ \neg(\neg F\lor\neg G)\end{matrix} \qquad \dfrac{\begin{matrix}[\neg G]^3\\ \vdots\\ \neg G\end{matrix}}{\neg F\lor\neg G}\lor I}{\bot}\neg E}{G}\bot\ (3)}{F\land G}\land I}{\bot}\neg E}{\neg F\lor\neg G}\bot\ (2)}{\neg(F\land G)\to(\neg F\lor\neg G)}\to I\ (1)$$

Exercise 4.2. [Classical Reasoning]
The intuitionistic version of natural deduction (NI) is obtained from the classical one by replacing the classical rule ($\perp$) with the rule $\dfrac{\perp}{F}$ ($\perp$).

Show that NI remains complete if we add either of the following rules to it:

- $\dfrac{}{F \vee \neg F}$ (law of excluded middle)
- $\dfrac{\neg\neg F}{F}$ (double negation elimination)

Solution:
We want to show that the classical ($\perp$) rule can be derived from either of the two other alternatives. Thus we assume that there is a proof of the form

$$\begin{array}{c} \neg F \\ \vdots \\ \perp \end{array}$$

and we need to show that we then can also prove F.

- $$\dfrac{\dfrac{}{F \vee \neg F}\text{(law of excluded middle)} \qquad [F]^1 \qquad \dfrac{\begin{array}{c}[\neg F]^1 \\ \vdots \\ \perp\end{array}}{F}\perp}{F}\vee E\ (1)$$

- $$\dfrac{\dfrac{\begin{array}{c}[\neg F]^1 \\ \vdots \\ \perp\end{array}}{\neg\neg F}\neg I\ (1)}{F}\text{(double negation elimination)}$$

Exercise 4.3. [Curry-Howard]
Let us explore the Curry-Howard correspondence in an interactive theorem prover.

Solution:
You can find the interactive code here. Check out one of our Isabelle courses if you like the idea of interactive theorem proving!

Homework 4.1. [More Natural Deduction] (++)

Prove the following formulas by natural deduction (as specified in the lecture):

1. $((A \to B) \to A) \to A$
2. $(\neg G \to F) \to (\neg F \to G)$
3. $\neg\neg\neg F \to \neg F$ (while using the $(\bot)$ rule from Exercise 4.2 and not the one defined in the lecture!)

Solution:

If you got stuck, reach out on Zulip. Or use this natural deduction prover for guidance.

Homework 4.2. [Substitution] (++)

Assume that there are proofs for $\vdash_N G \to G'$ and $\vdash_N G' \to G$. Construct the proof for $\vdash_N F[G/A] \to F[G'/A]$.

Solution:

We prove the claim by induction on F. We strengthen the IH and also prove $\vdash_N F[G'/A] \to F[G/A]$. We only show the cases for the functional complete subset $\{\neg, \wedge\}$ (the other cases are similar):

<u>Case A_i</u>: The case $A_i \neq A$ is trivial. If $A_i = A$, we have $A_i[G/A] = G$ and $A_i[G'/A] = G'$ and $G \to G'$ as well as $G' \to G$ by assumption, so we are done.

<u>Case $\neg F$</u>: Note that $(\neg F)\sigma = \neg F\sigma$ for any substitution σ. We then have

$$\dfrac{\dfrac{\dfrac{\big[(\neg F)[G/A]\big]^1 \qquad \dfrac{\dfrac{}{F[G'/A] \to F[G/A]}(IH) \qquad \big[F[G'/A]\big]^2}{F[G/A]}(\to E)}{\bot}(\to E)}{(\neg F)[G'/A]}(\to I)_2}{(\neg F)[G/A] \to (\neg F)[G'/A]}(\to I)_1$$

The other direction is symmetric.

<u>Case $F \wedge G$</u>: Note that $(F \wedge G)\sigma = F\sigma \wedge G\sigma$ for any substitution σ. We then have

$$\dfrac{\dfrac{\dfrac{\dfrac{}{F[G/A] \to F[G'/A]}(IH) \qquad \dfrac{\big[F[G/A] \wedge G[G/A]\big]^1}{F[G/A]}(\wedge E_1)}{F[G'/A]}(\to E) \qquad \dfrac{\ldots\text{analagous}}{G[G'/A]}(\to E)}{F[G'/A] \wedge G[G'/A]}(\wedge I)}{F[G/A] \wedge G[G/A] \to F[G'/A] \wedge G[G'/A]}(\to I)_1$$

The other direction is symmetric.

Homework 4.3. [Glivenko's Theorem] (+++)
On exercise sheet 3, you already discovered that although some classical laws are not derivable in intuitionistic logic, their doubly negated variants indeed are derivable.

Here's the complete story: Glivenko's Theorem states that F is valid classically if and only if $\neg\neg F$ is valid intuitionistically. Prove Glivenko's Theorem, i.e. show $\vdash_N F \iff \vdash_{NI} \neg\neg F$.

Solution:
The direction from right to left is trivial: all rules in NI are valid in N and double negation elimination is valid in N as well as shown in the tutorial.

To show the other direction, we proceed by induction on the derivation of $\vdash_N F$. You can find some, but not all cases (the missing ones are pretty similiar), on the final page of the sheet.

> Taking away the tertium non datur from the mathematician would be about the same as if one would forbid the telescope to the astronomer or the use of his fists to the boxer.
>
> — David Hilbert

Case $\dfrac{F \quad G}{F \wedge G}(\wedge I)$ IH: $\vdash_{NI} \neg\neg F$, $\vdash_{NI} \neg\neg G$. To show: $\vdash_{NI} \neg\neg(F \wedge G)$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{(IH)}
\UnaryInfC{$\neg\neg F$}
\AxiomC{}
\RightLabel{(IH)}
\UnaryInfC{$\neg\neg G$}
\AxiomC{$[\neg(F\wedge G)]^1$}
\AxiomC{$[F]^2$}
\AxiomC{$[G]^3$}
\RightLabel{$(\wedge I)$}
\BinaryInfC{$F \wedge G$}
\RightLabel{$(\rightarrow E)$}
\BinaryInfC{$\bot$}
\RightLabel{$(\rightarrow I)3$}
\UnaryInfC{$\neg G$}
\RightLabel{$(\rightarrow E)$}
\BinaryInfC{$\bot$}
\RightLabel{$(\rightarrow I)2$}
\UnaryInfC{$\neg F$}
\RightLabel{$(\rightarrow E)$}
\BinaryInfC{$\bot$}
\RightLabel{$(\rightarrow I)1$}
\UnaryInfC{$\neg\neg(F \wedge G)$}
\end{prooftree}
$$

Case $\dfrac{F_1 \wedge F_2}{F_i}(\wedge E_i)$ IH: $\vdash_{NI} \neg\neg(F_1 \wedge F_2)$ To Show: $\vdash_{NI} \neg\neg F_i$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{(IH)}
\UnaryInfC{$\neg\neg(F_1 \wedge F_2)$}
\AxiomC{$[\neg F_i]^1$}
\AxiomC{$[F_1 \wedge F_2]^2$}
\RightLabel{$(\wedge E_i)$}
\UnaryInfC{F_i}
\BinaryInfC{$\bot$}
\RightLabel{$(\rightarrow I)2$}
\UnaryInfC{$\neg(F_1 \wedge F_2)$}
\RightLabel{$(\rightarrow E)$}
\BinaryInfC{$\bot$}
\RightLabel{$(\rightarrow I)1$}
\UnaryInfC{$\neg\neg F_i$}
\end{prooftree}
$$

Case $\dfrac{\begin{matrix}[F]\\ \vdots \\ G\end{matrix}}{F \rightarrow G}(\rightarrow I)$ IH: $F \vdash_{NI} \neg\neg G$ To Show: $\vdash_{NI} \neg\neg(F \rightarrow G)$

$$
\begin{prooftree}
\AxiomC{$[\neg(F\rightarrow G)]^1$}
\AxiomC{$[F]^2$}
\noLine
\UnaryInfC{$\vdots$}
\RightLabel{(IH+2)}
\UnaryInfC{$\neg\neg G$}
\AxiomC{$[\neg(F\rightarrow G)]^1$}
\AxiomC{$[G]^3$}
\RightLabel{$(\rightarrow I)$}
\UnaryInfC{$F \rightarrow G$}
\BinaryInfC{$\bot$}
\RightLabel{$(\rightarrow I)3$}
\UnaryInfC{$\neg G$}
\RightLabel{$(\rightarrow E)$}
\BinaryInfC{$\bot$}
\RightLabel{$(\bot)$}
\UnaryInfC{G}
\RightLabel{$(\rightarrow I)2$}
\UnaryInfC{$F \rightarrow G$}
\RightLabel{$(\rightarrow E)$}
\BinaryInfC{$\bot$}
\RightLabel{$(\rightarrow I)1$}
\UnaryInfC{$\neg\neg(F \rightarrow G)$}
\end{prooftree}
$$

Case $\dfrac{\begin{matrix}[\neg F]\\ \vdots \\ \bot\end{matrix}}{F}(\bot)$ IH: $\neg F \vdash_{NI} \neg\neg\bot$ To show: $\vdash_{NI} \neg\neg F$

$$
\begin{prooftree}
\AxiomC{$[\neg F]^1$}
\noLine
\UnaryInfC{$\vdots$}
\RightLabel{(IH)}
\UnaryInfC{$\neg\neg\bot$}
\AxiomC{$[\bot]^2$}
\RightLabel{$(\rightarrow I)2$}
\UnaryInfC{$\neg\bot$}
\BinaryInfC{$\bot$}
\RightLabel{$(\rightarrow I)1$}
\UnaryInfC{$\neg\neg F$}
\end{prooftree}
$$