Logic Exercises

Technical University of Munich
Chair for Logic and Verification

Prof. Tobias Nipkow
Kevin Kappelmann

SS 2022 — Exercise Sheet 5 — 27.05.2022

Exercise 5.1. **[A Family of Formulas]**
Show that the following schema has a proof in natural deduction for all $n \geq 1$:

$$P_n = ((A_1 \wedge (A_2 \wedge (\cdots \wedge A_n) \cdots) \to B) \to (A_1 \to (A_2 \to (\cdots (A_n \to B) \cdots))))$$

Exercise 5.2. **[From Sequent Calculus to Natural Deduction]**
How can we construct a natural deduction proof $\Gamma \vdash_{\mathrm{N}} \bigvee \Delta$ from a sequent calculus proof $\Gamma \Rightarrow \Delta$?

Exercise 5.3. **[Hilbert Calculus]**
Prove the following formula with a linear proof in Hilbert calculus: $(F \wedge G) \to (G \wedge F)$

Hint: Use the deduction theorem.

Exercise 5.4. **[From Hilbert Calculus to Natural Deduction]**
Prove: if $\Gamma \vdash_H F$ then $\Gamma \vdash_N F$.

Homework 5.1. [Small Hilbert] (++)
In the lecture, Hilbert calculus for propositional logic was introduced by means of nine axioms. However, the following three axioms are already sufficient:

A1 $F \to (G \to F)$

A2 $(F \to G \to H) \to (F \to G) \to F \to H$

A10 $(\neg F \to \neg G) \to (G \to F)$

Derive the following statement from the axioms above with the help of $\to_E$:

$$\neg(F \to F) \to G$$

Optional: In fact, Meredith showed that all that is needed is one single axiom:

$$((((A \to B) \to (\neg C \to \neg D)) \to C) \to E) \to ((E \to A) \to (D \to A))$$

Try to derive some axiom of your choice presented in the lecture in Meredith's system.

Homework 5.2. [From Sequent Calculus to Natural Deduction: Reloaded] (++)
In Exercise 5.2, we constructed a natural deduction proof $\Gamma \vdash_{\mathrm{N}} \bigvee \Delta$ from a sequent calculus proof of $\Gamma \Rightarrow \Delta$. That construction created classical proofs because it required the use of the $(\bot)$ rule.

Let us consider yet another restricted Sequent Calculus called "G3c". In G3c, we have $\Delta = \{F\}$, that is the succedent always contains exactly one formula. Here are the axioms:

Ax $P, \Gamma \Rightarrow P$ (P atomic) L$\bot$ $\bot, \Gamma \Rightarrow A$

$$\text{L}\wedge\ \frac{A, B, \Gamma \Rightarrow C}{A \wedge B, \Gamma \Rightarrow C} \qquad \text{R}\wedge\ \frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}$$

$$\text{L}\vee\ \frac{A, \Gamma \Rightarrow C \qquad B, \Gamma \Rightarrow C}{A \vee B, \Gamma \Rightarrow C} \qquad \text{R}\vee\ \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_0 \vee A_1}\ (i = 0, 1)$$

$$\text{L}\to\ \frac{A \to B, \Gamma \Rightarrow A \qquad B, \Gamma \Rightarrow C}{A \to B, \Gamma \Rightarrow C} \qquad \text{R}\to\ \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \to B}$$

Give a direct construction that transforms a G3c proof $\Gamma \Rightarrow F$ into a natural deduction proof $\Gamma \vdash_{\mathrm{N}} F$ without using the $(\bot)$ rule. You are, however, allowed to use the intuitionistic rule $\bot \vdash_N F$; call it $(\bot E)$.

Homework 5.3. **[Simulating Truth Tables]** (+)
In the lecture, the following lemma was discussed:

Let $\text{atoms}(F) \subseteq \{A_1, \ldots, A_n\}$. Then we can construct a proof $A_1^{\mathcal{A}}, \ldots, A_n^{\mathcal{A}} \vdash_{\mathrm{N}} F^{\mathcal{A}}$.

Recall the definition of $F^{\mathcal{A}}$:

$$F^{\mathcal{A}} = \begin{cases} F, & \text{if } \mathcal{A}(F) = 1 \\ \neg F, & \text{otherwise} \end{cases}$$

The proof proceeded by induction on F. The cases for atomic formulas as well as implication were shown in the lecture. Prove the cases for negation and disjunction!

Simplicity is the ultimate sophistication.

— William Gaddis