# Logic Exercises

Technical University of Munich
Chair for Logic and Verification

Prof. Tobias Nipkow
Kevin Kappelmann

SS 2022 | Exercise Sheet 9 | 24.06.2022

**Exercise 9.1.** **[Wait, What?]**

1. Resolution for first-order logic is sound and complete.
2. The satisfiability and validity problems for first-order logic are undecidable.

How do you reconcile these two facts? Write down the definitions of all above used logical terminology (sound, complete, undecidable, etc.) and discuss the consequences of above facts.

**Exercise 9.2.** **[Do You Even Lifting Lemma?]**
Consider the following resolution:

$$\{P(x,y), P(f(a),z)\} \quad\xrightarrow{[f(a)/x]\,[g(b)/y]\,[g(b)/z]}\quad \{P(f(a),g(b))\}$$

$$\{\neg P(f(x),g(y)), Q(x,y)\} \quad\xrightarrow{[a/x]\,[b/y]}\quad \{\neg P(f(a),g(b)), Q(a,b)\}$$

$$\frac{\{P(f(a),g(b))\} \qquad \{\neg P(f(a),g(b)), Q(a,b)\}}{\{Q(a,b)\}}$$

Follow the proof of the Lifting Lemma and find out which (predicate logic) resolution step is constructed from this.

**Exercise 9.3.** **[Green Dragon Children Are Cute Unless You Have to Fight Them]**
Express the following facts by formulas in predicate logic.

1. Every dragon is happy if all its children can fly.
2. Green dragons can fly.
3. A dragon is green if it is a child of at least one green dragon.

Prove by resolution that the conjunction of these three statements implies the following: all green dragons are happy.

**Exercise 9.4.** **[Justice > Equity > Equality]**
We consider how to model equality in predicate logic. In the lecture slides, the following rule for congruence is used:

$$\forall x_1, \ldots, x_n, y.\ Eq(x_i, y) \rightarrow Eq(f(x_1, \ldots, x_i, \ldots, x_n), f(x_1, \ldots, y, \ldots, x_n)).$$

We can also write this as an inference rule:

$$\frac{Eq(x_i, y)}{Eq(f(x_1, \ldots, x_i, \ldots, x_n), f(x_1, \ldots, y, \ldots, x_n))}$$

Assume that this rule is replaced by:

$$\frac{Eq(x_1, y_1) \qquad \cdots \qquad Eq(x_n, y_n)}{Eq(f(x_1, \ldots, x_n), f(y_1, \ldots, y_n))}$$

Reflexivity, symmetry and transitivity stay unchanged. Show that the modified set of rules is equivalent to the set of rules from the lecture.

**Homework 9.1. [Blackbox Proving]** (++)
Assume you are given an algorithm $\mathcal{A}$ operating on first-order CNF formulas such that whenever the resolution calculus produces a resolvent $R$ from clauses $C_1$ and $C_2$, $\mathcal{A}$ produces a clause $R' \subseteq R$.

Prove that $\mathcal{A}$ is refutationally complete.

**Homework 9.2. [Binary Resolution]** (+++)
In the resolution procedure as defined in the lecture slides, we can unify arbitrarily many literals from two clauses. Consider a modified resolution procedure where exactly one literal is picked in each clause ("binary resolution"). We add a new rule ("factoring"): for a clause $C = \{L_1, \dots, L_n\}$, where $\{L_i, L_j\}$ can be unified using an mgu $\sigma$ with $i \neq j$, add another clause $C' = (C \setminus L_i)\sigma$.

For example, given the clause

$$C = \{\neg W(x), \neg W(f(y)), T(x, y), \neg W(f(c))\}$$

we can apply the factoring rule as follows:

$$L_1 = \neg W(x), L_2 = \neg W(f(y)), \sigma = \{x \mapsto f(y)\}, C' = \{\neg W(f(y)), T(f(y), y), \neg W(f(c))\}$$

1. Prove that binary resolution without factoring is incomplete.
2. Prove that binary resolution with factoring is complete.

**Homework 9.3. [Equality Elimination]** (+)
Show with resolution that: $f(f(f(a))) = a \to \big(f(f(a)) = a \to f(a) = a\big)$ is valid. First, remove equality based on the procedure from the lecture. Then perform resolution.

**Homework 9.4. [(Bonus) A Barbarian Bavarian Barber Walks Into a Barber]** (++)
You can solve this exercise if you need more practice with FOL-resolution; but you will not miss anything if you do not – there is no new content in this exercise.

Consider the signature $\{B, S\}$ where $B$ is a unary predicate expressing that an element represents a barber; while $S(x, y)$ indicates that "$x$ shaves $y$".

1. Axiomatize the statements:
    - Persons who do not shave themselves are shaven by all barbers.
    - No barber shaves persons who shave themselves.
2. Show by resolution that the fact that no barbers exist is a consequence of the two statements above.

> The past was erased,
> the erasure was forgotten,
> the lie became the truth.
>
> — George Orwell