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Chair for Logic and Verification

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SS 2022 Exercise Sheet 10 01.07.2022

Exercise 10.1. [$\exists^*\forall^*$ **with Equality**]
Show that unsatisfiability of formulas from the $\exists^*\forall^*$ fragment with equality is decidable.

Exercise 10.2. [$\exists^*\forall^2\exists^*$]
Show how to reduce deciding unsatisfiability of formulas from the $\exists^*\forall^2\exists^*$-fragment to deciding unsatisfiability of formulas from the $\forall^2\exists^*$-fragment.

Exercise 10.3. [**Sequent Calculus**]
Prove the following formulas in sequent calculus:

1. $\neg\exists x P(x) \to \forall x \neg P(x)$
2. $(\forall x(P \vee Q(x))) \to (P \vee \forall x Q(x))$

Exercise 10.4. [**Can't Touch This**]
Let $\mathcal{A}, \mathcal{B}$ be structures over the same language with universes A and B, respectively. We say that $\mathcal{A}, \mathcal{B}$ are *isomorphic* if there is a bijection $i : A \to B$ which preserves the interpretation of all symbols, that is:

1. $i(c^{\mathcal{A}}) = c^{\mathcal{B}}$, for all constants c
2. $i(f^{\mathcal{A}}(a_1, \ldots, a_n)) = f^{\mathcal{B}}(i(a_1), \ldots, i(a_n))$, for all functions f and $a_1, \ldots, a_n \in A$
3. $P^{\mathcal{A}}(a_1, \ldots, a_n) \iff P^{\mathcal{B}}(i(a_1), \ldots, i(a_n))$, for all predicates P and $a_1, \ldots, a_n \in A$

Let $\mathcal{N}$ be the standard model of the natural numbers. Assume you are given a countable first-order axiomatisation T of $\mathcal{N}$. Show that there is another model $\mathcal{N}'$ of T that is not isomorphic to $\mathcal{N}$.

Homework 10.1. **[FOL without Function Symbols]** (+++)
Describe an algorithm that transforms any formula F (in FOL with equality) into an equisatisfiable formula F' (in FOL with equality) that does not use function symbols. Do not forget to deal with constants, i.e. functions with arity 0.

Apply your algorithm to the formula $F \coloneqq \forall xy.\, R(f(x,y)) \wedge P(c, g(f(x,y)))$.

Homework 10.2. **[Undefinability of Finiteness]** (++)
In the following, given a structure $\mathcal{A}$, we write $A \coloneqq U^{\mathcal{A}}$.

1. Give a countable set of sentences S_I such that for any structure $\mathcal{A}$, $\mathcal{A} \models S_I$ if and only if A has infinitely many elements.
2. Show that there cannot be a countable set of sentences S_F such that for any structure $\mathcal{A}$, $\mathcal{A} \models S_F$ if and only if A has finitely many elements.

Homework 10.3. **[Sequent Calculus]** (++)
Prove the following statements using sequent calculus if they are valid, or give a countermodel otherwise.

1. $\neg\forall x \exists y \forall z (\neg P(x,z) \wedge P(z,y))$
2. $\forall x \forall y \forall z (P(x,x) \wedge (P(x,y) \wedge P(y,z) \rightarrow P(x,z)))$

Homework 10.4. **[Miniscoping]** (++)
In the lecture, we proved that deciding unsatisfiability of monadic FOL formulas can be reduced to deciding unsatisfiability of formulas from the $\exists^*\forall^*$ fragment by using miniscoping.

Prove the lemma that after miniscoping, no nested quantifiers remain.

> Logic is in the eye of the logician.
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> — Gloria Steinem