Logic Exercises

Technical University of Munich
Chair for Logic and Verification

Prof. Tobias Nipkow
Kevin Kappelmann

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Exercise 10.1. [$\exists^*\forall^*$ **with Equality**]
Show that unsatisfiability of formulas from the $\exists^*\forall^*$ fragment with equality is decidable.

Solution:
Applying the reduction of equality to non-equality from the lecture only inserts some (isolated) $\forall$-quantifiers, thus preserving the $\exists^*\forall^*$-fragment.

Exercise 10.2. [$\exists^*\forall^2\exists^*$]
Show how to reduce deciding unsatisfiability of formulas from the $\exists^*\forall^2\exists^*$-fragment to deciding unsatisfiability of formulas from the $\forall^2\exists^*$-fragment.

Solution:
Using skolemization for the outer existential quantifiers preserves satisfiability, and replaces variables by skolem constants, i.e., introduces no function symbols of arity > 0. The resulting formula is obviously in the $\forall^2\exists^*$-fragment.

Exercise 10.3. [**Sequent Calculus**]
Prove the following formulas in sequent calculus:

1. $\neg\exists x P(x) \to \forall x \neg P(x)$
2. $(\forall x(P \lor Q(x))) \to (P \lor \forall x Q(x))$

Solution:

1.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P(y) \Rightarrow \exists x P(x), P(y)}}{\Rightarrow P(y), \exists x P(x), \neg P(y)}\ \neg R}{\Rightarrow \exists x P(x), \neg P(y)}\ \exists R}{\Rightarrow \exists x P(x), \forall x \neg P(x)}\ \forall R}{\neg\exists x P(x) \Rightarrow \forall x \neg P(x)}\ \neg L}{\Rightarrow \neg\exists x P(x) \to \forall x \neg P(x)}\ \to R$$

2.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(\forall x(P \lor Q(x))), P \Rightarrow P, Q(x)}\ Ax \qquad \dfrac{}{(\forall x(P \lor Q(x))), Q(x) \Rightarrow P, Q(x)}\ Ax}{(\forall x(P \lor Q(x))), (P \lor Q(x)) \Rightarrow P, Q(x)}\ \lor L}{\forall x(P \lor Q(x)) \Rightarrow P, Q(x)}\ \forall L}{\forall x(P \lor Q(x)) \Rightarrow P, \forall x Q(x)}\ \forall R}{\forall x(P \lor Q(x)) \Rightarrow P \lor \forall x Q(x)}\ \lor R}{\Rightarrow (\forall x(P \lor Q(x))) \to (P \lor \forall x Q(x))}\ \to R$$

Exercise 10.4. [Can't Touch This]
Let $\mathcal{A}, \mathcal{B}$ be structures over the same language with universes A and B, respectively. We say that $\mathcal{A}, \mathcal{B}$ are *isomorphic* if there is a bijection $i : A \to B$ which preserves the interpretation of all symbols, that is:

1. $i(c^{\mathcal{A}}) = c^{\mathcal{B}}$, for all constants c
2. $i(f^{\mathcal{A}}(a_1, \ldots, a_n)) = f^{\mathcal{B}}(i(a_1), \ldots, i(a_n))$, for all functions f and $a_1, \ldots, a_n \in A$
3. $P^{\mathcal{A}}(a_1, \ldots, a_n) \iff P^{\mathcal{B}}(i(a_1), \ldots, i(a_n))$, for all predicates P and $a_1, \ldots, a_n \in A$

Let $\mathcal{N}$ be the standard model of the natural numbers. Assume you are given a countable first-order axiomatisation T of $\mathcal{N}$ over some signature Σ. Show that there is another model $\mathcal{N}'$ of T that is not isomorphic to $\mathcal{N}$.

Solution:
Let c be a fresh constant. Consider the set of sentences $T' \coloneqq T \cup \{c \neq n \mid n \in \mathbb{N}\}$. Intuitively, c denotes an element that is different from all natural numbers. Note that T' is countable.

We now apply compactness: Take a finite subset S of T'. S contains only finitely many sentences of the shape $c \neq n$. Let $m \coloneqq 1 + \max\{n \mid n = 0 \vee (c \neq n) \in S\}$. Extend $\mathcal{N}$ by adding the constant c and interpret it by m. Then $\mathcal{N} \models S$.

Hence, by the compactness theorem, there is $\mathcal{N}'$ with $\mathcal{N}' \models T'$. Thus, $\mathcal{N}' \models T$ but $\mathcal{N}'$ contains an element $c^{\mathcal{N}'}$ that is different from all natural numbers and hence cannot be isomorphic to $\mathcal{N}$.

To see that $\mathcal{N}'$ is not isomorphic to $\mathcal{N}$ more formally, assume there is an isomorphism i from $\mathcal{N}'$ to $\mathcal{N}$. Let $n \coloneqq i(c^{\mathcal{N}'}) \in \mathcal{U}^{\mathcal{N}}$. Then $c^{\mathcal{N}'} = n^{\mathcal{N}'} \overset{i \text{ is iso.}}{\iff} i(c^{\mathcal{N}'}) = i(n^{\mathcal{N}'}) \iff n = i(n^{\mathcal{N}'}) \overset{i \text{ is iso.}}{\iff} n = n$. However, $c^{\mathcal{N}'} = n^{\mathcal{N}'}$ is false and $n = n$ true, contradiction.

Homework 10.1. [FOL without Function Symbols] (+++)
Describe an algorithm that transforms any formula F (in FOL with equality) into an equisatisfiable formula F' (in FOL with equality) that does not use function symbols. Do not forget to deal with constants, i.e. functions with arity 0.

Apply your algorithm to the formula $F \coloneqq \forall xy.\, R(f(x,y)) \wedge P(c, g(f(x,y)))$.

Solution:
Idea: functions can be modelled as relations satisfying some additional properties (totality + right-uniqueness).

1. For any function f/n, introduce a fresh predicate P_f of arity $n+1$.
2. Add the following conjunct for each new predicate: $\forall x_1 \cdots x_n.\, \exists y \left(P_f(x_1,\ldots,x_n,y) \wedge \forall z.\, (P_f(x_1,\ldots,x_n,z) \rightarrow y = z)\right)$
3. Iteratively replace all innermost occurences of $f(x_1,\ldots,x_n)$ in F by some fresh, universally bounded variable z and add the premise $P_f(x_1,\ldots,x_n,z)$.

Example, step by step, excluding the new predicates' conjuncts:

1. $\forall x,y,z_1.\, (P_c(z_1) \rightarrow R(f(x,y)) \wedge P(z_1, g(f(x,y))))$
2. $\forall x,y,z_1,z_2.\, (P_c(z_1) \rightarrow P_f(x,y,z_2) \rightarrow R(z_2) \wedge P(z_1, g(z_2)))$
3. $\forall x,y,z_1,z_2,z_3.\, (P_c(z_1) \rightarrow P_f(x,y,z_2) \rightarrow P_g(z_2,z_3) \rightarrow R(z_2) \wedge P(z_1,z_3))$

Clearly, by interpreting each P_f by $P_f \coloneqq \{(e_1,\ldots,e_n,e) \mid f(e_1,\ldots,e_n) = e\}$, each model of F can be transformed to a model of F'. Conversely, if F' has a model, then each P_f can be used to interpret the function f, allowing us to construct a model for F.

Homework 10.2. [Undefinability of Finiteness] (++)
In the following, given a structure $\mathcal{A}$, we write $A \coloneqq U^{\mathcal{A}}$.

1. Give a countable set of sentences S_I such that for any structure $\mathcal{A}$, $\mathcal{A} \models S_I$ if and only if A has infinitely many elements.
2. Show that there cannot be a countable set of sentences S_F such that for any structure $\mathcal{A}$, $\mathcal{A} \models S_F$ if and only if A has finitely many elements.

Solution:

1. Let $F_n \coloneqq \exists x_1,\ldots,x_n.\, \bigwedge_{1 \le i < j \le n}^{n} x_i \neq x_j$ and $S_I \coloneqq \{F_n \mid n \in \mathbb{N}_+\}$. If A is infinite, then $\mathcal{A} \models F_n$ for each n and hence $\mathcal{A} \models S_I$. If $|A| \coloneqq n \in \mathbb{N}_+$, then $\mathcal{A} \not\models F_{n+1}$ and hence $\mathcal{A} \not\models S_I$.
2. Assume there is such a set S_F. Consider the set $S \coloneqq S_F \cup S_I$. Take a finite subset $T \subset S$. Let $m \coloneqq \max\{n \mid n = 1 \vee F_n \in T\}$. Let $\mathcal{A}$ be an arbitrary structure for S_F of size greater than m. Then $\mathcal{A} \models F_i$ for all $1 \le i \le m$ and $\mathcal{A} \models S_F$. Hence, $\mathcal{A} \models T$. Thus, by compactness, S has a model $\mathcal{M}$. Then $\mathcal{M} \models S_F$ and hence $U^{\mathcal{M}}$ is finite by assumption, but also $\mathcal{M} \models S_I$ and hence $U^{\mathcal{M}}$ is infinite by the previous exercise. Contradiction!

Homework 10.3. **[Sequent Calculus]** (++)
Prove the following statements using sequent calculus if they are valid, or give a countermodel otherwise.

1. $\neg\forall x\exists y\forall z(\neg P(x,z) \land P(z,y))$
2. $\forall x\forall y\forall z(P(x,x) \land (P(x,y) \land P(y,z) \to P(x,z)))$

Homework 10.4. **[Miniscoping]** (++)
In the lecture, we proved that deciding unsatisfiability of monadic FOL formulas can be reduced to deciding unsatisfiability of formulas from the $\exists^*\forall^*$ fragment by using miniscoping.

Prove the lemma that after miniscoping, no nested quantifiers remain.

Solution:
We prove by induction on the structure of the formula that after miniscoping, for each subformula of the form $Qx.\ F$, F is a disjunction of literals if $Q = \forall$ and conjunction of literals if $Q = \exists$ and each literal contains x free.

The only interesting cases are the quantifier cases. Assume we have a formula of the form $\exists xF$ such that no miniscoping rules are applicable. By the induction hypothesis, below all quantifiers in F, there are only disjunctions/conjunctions of literals containing the bound variable.

As no miniscoping rules are applicable, F must be a conjunction of literals and quantified formulas such that each conjunct contains x free. So assume F contains a quantified formula, that is $F = \cdots \land Qy.F' \land \cdots$. By the induction hypothesis, F' is a disjunction/conjunction of literals, each literal containing y free. However, as we are in the monadic fragment, a literal can contain at most one free variable. Thus, F' cannot contain x free, which is a contradiction to F containing quantifiers. Thus, F only contains literals and hence has the desired shape.

The case for $\forall xF$ is similar.

> Logic is in the eye of the logician.
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> — Gloria Steinem