# Logic Exercises

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**Exercise 11.1. [This Is Unnatural]**
Let $\mathcal{N}$ be the standard model of the natural numbers. In tutorial exercise 10.4, we proved that any countable axiomatisation $T$ of $\mathcal{N}$ (i.e. $Cn(T) = Th(\mathcal{N})$) over some signature $\Sigma$ admits another, non-isomorphic model $\mathcal{N}'$ (in particular, $Cn(T) = Th(\mathcal{N}')$).

Prove that $\mathcal{N}'$ contains not only one, but infinitely many non-standard natural numbers $d_n \in U^{\mathcal{N}'}$, i.e. $\mathcal{N}'[d_n/x] \models x \neq m$ for all $n, m \in \mathbb{N}$ and $\mathcal{N}'[d_n/x, d_m/y] \models x \neq y$ for all $n, m \in \mathbb{N}$ with $n \neq m$.

**Solution:**
First note that $Th(\mathcal{N}) = Cn(T) = Th(\mathcal{N}')$. Thus, $\mathcal{N} \models S \iff \mathcal{N}' \models S$ for any $\Sigma$-sentence $S$.

Now consider $d_n \coloneqq (c+n)^{\mathcal{N}'}$ for any $n \in \mathbb{N}$. We show that $\mathcal{N}' \models c+n \neq m$ for any $m \in \mathbb{N}$. We have $\mathcal{N} \models \forall k.(k+n = m \rightarrow k = m-n)$. Hence $\mathcal{N}' \models \forall k.(k+n = m \rightarrow k = m-n)$, and thus $\mathcal{N}' \models c+n = m \rightarrow c = m-n$. For the sake of contradiction, assume $\mathcal{N}' \models c+n = m$. Then by the previous, $\mathcal{N}' \models c = m-n$. But also $\mathcal{N}' \models c \neq k$ for all $k \in \mathbb{N}$ by construction of $\mathcal{N}'$, contradiction.

It remains to prove that $\mathcal{N}' \models c+n \neq c+m$ for $n \neq m$. Since $\mathcal{N} \models \forall k.\ (k+n \neq k+m \leftrightarrow n \neq m)$ and $n \neq m$ by assumption, we get $\mathcal{N}' \models c+n \neq c+m$.

**Exercise 11.2. [Decidability of Complete Theories]**
Assume $S$ is finitely axiomatizable and complete, i.e. $F \in S$ or $\neg F \in S$ for any sentence $F$.

1. Given only the axiomatization of $S$, give a procedure deciding whether $S \models F$ for any sentence $F$.
2. Can you obtain a similar result when the assumption is that the axiom system is only *recursively enumerable*?

**Solution:**

1. Let $M$ be the set of axioms. Run resolution on $M \wedge F$ and $M \wedge \neg F$ in parallel. If $F \notin S$, then $M \wedge F \vdash \square$ and the first resolution terminates. If $F \in S$, then $M \wedge \neg F \vdash \square$ and the second resolution terminates.
2. Yes, by compactness. Enumerate all finite subsets of the axiom set and execute the resolution calls in a dovetailing approach.

**Exercise 11.3. [One Finite, All Finite]**
Show that if a theory is finitely axiomatizable, any countable axiomatization of it has a finite subset that axiomatizes the same theory. In other words, if $Cn(\Gamma) = Cn(\Delta)$ with $\Gamma$ countable and $\Delta$ finite, then there is a finite $\Gamma' \subseteq \Gamma$ with $Cn(\Gamma') = Cn(\Gamma)$. Can you also obtain $\Gamma'$ effectively?

**Solution:**
Let us identify $\Delta$ as the formula $\bigwedge_{F \in \Delta} F$. It suffices to find a finite subset $\Gamma' \subseteq \Gamma$ that axiomatizes $Cn(\Delta)$. For this, it is sufficient to find $\Gamma' \subseteq \Gamma$ such that $\Gamma' \models \Delta$, which is equivalent to $\Gamma' \cup \{\neg\Delta\}$ being unsatisfiable.

We know that $\Gamma \cup \{\neg\Delta\}$ is unsatisfiable because $\Gamma$ axiomatizes $Cn(\Delta)$. By compactness, there must be a finite subset that is unsatisfiable. We can find this subset by enumerating all finite subsets $\Gamma' \subseteq \Gamma$ and running resolution on $\Gamma', \neg\Delta$.

**Exercise 11.4. [Natural Deduction]**
Prove the following formula using natural deduction.

$$\neg(\forall x(\exists y(\neg P(x) \wedge P(y))))$$

**Solution:**

$$\dfrac{\forall E\ \dfrac{[\forall x\exists y(\neg P(x)\wedge P(y))]}{\exists y(\neg P(x_1)\wedge P(y))} \qquad \dfrac{\dfrac{[\neg P(x_1)\wedge P(y_1)]}{P(y_1)}\ \wedge E_2 \qquad \dfrac{\forall E\ \dfrac{[\forall x\exists y(\neg P(x)\wedge P(y))]}{\exists y(\neg P(y_1)\wedge P(y))} \qquad \dfrac{[\neg P(y_1)\wedge P(y_2)]}{\neg P(y_1)}\ \wedge E_1}{\neg P(y_1)}\ \exists E}{\bot}\ \neg E}{\dfrac{\bot}{\neg(\forall x(\exists y(\neg P(x)\wedge P(y))))}\ \neg I}\ \exists E$$

**Homework 11.1.** **[Counterexamples from Sequent Calculus]** (++)

Consider the statement $\forall x(P(x) \to \neg P(f(x)))$.

1. What happens when trying to prove the validity of this formula in sequent calculus?
2. How can we derive a countermodel from the proof tree?
3. Is there a smaller countermodel?

**Solution:**

The proof tree gets stuck:

$$\dfrac{\dfrac{\dfrac{P(y), P(f(y)) \Rightarrow}{P(y) \Rightarrow \neg P(f(y))}\,\neg R}{\Rightarrow P(y) \to \neg P(f(y))}\,\to R}{\Rightarrow \forall x\,(P(x) \to \neg P(f(x)))}\,\forall R$$

As in the lecture, we can create a countermodel $\mathcal{A}$: Let $U_\mathcal{A}$ be the set of all terms over $y, f(\cdot)$, set $y^\mathcal{A} := y$, $f^\mathcal{A}(t) := f(t^\mathcal{A})$, and $P^\mathcal{A} := \{y, f(y)\}$. Then $\mathcal{A} \models P(y)$ and $\mathcal{A} \models P(f(y))$ and hence $\mathcal{A} \not\models \forall x\,(P(x) \to \neg P(f(x)))$. Note that $\mathcal{A}$ is infinite, but there are countermodels with just two elements $\{a, b\}$: Set $f(a) := b$, $f(b) := b$, $P(a)$ and $P(b)$. Then $P(a)$ and $P(f(a)) = P(b)$.

**Homework 11.2.** **[Natural Deduction]** (++)

Prove the following statements using natural deduction.

1. $\neg\forall x\exists y\forall z(\neg P(x,z) \land P(z,y))$
2. $\exists x(P(x) \to \forall x P(x))$

**Solution:**

Here's an outline for the second task. You can ask for further hints on Zulip.

$$\dfrac{\text{EXCL. MIDDLE}\ \dfrac{Exercise(EX)}{\forall x P(x) \lor \neg\forall x P(x)} \quad \dfrac{\dfrac{[\forall x P(x)]_1}{P(x_0) \to \forall x P(x)}\,\to I}{\exists x(P(x) \to \forall x P(x))}\,\exists I \quad \dfrac{\dfrac{\begin{array}{c}[\neg\forall x P(x)]_1\\ \vdots\end{array}}{\exists x.\neg P(x)}\,(\mathrm{EX}) \quad \dfrac{\dfrac{\dfrac{\dfrac{[P(x_0)]_3 \quad [\neg P(x_0)]_2}{\bot}\,\neg E}{\forall x P(x)}\,\bot}{P(x_0) \to \forall x P(x)}\,(\to I)_3}{\exists x(P(x) \to \forall x P(x))}\,\exists I}{\exists x(P(x) \to \forall x P(x))}\,(\exists E)_2}{\exists x(P(x) \to \forall x P(x))}\,(\lor E)_1$$

**Homework 11.3. [Closure Operator]** (+)

Show that $Cn$ is a closure operator, i.e. $Cn$ fulfills the following properties:

- $S \subseteq Cn(S)$
- if $S \subseteq S'$ then $Cn(S) \subseteq Cn(S')$
- $Cn(Cn(S)) = Cn(S)$

**Solution:**

In the following, suppose $S$, $S'$ are sets of $\Sigma$-sentences and $F$ is a $\Sigma$-sentence.

- $F \in S \Longrightarrow S \models F \Longrightarrow F \in Cn(S)$
- $F \in Cn(S) \Longrightarrow S \models F \Longrightarrow S' \models F \Longrightarrow F \in Cn(S')$
- From the first property, we get $Cn(S) \subseteq Cn(Cn(S))$. For the other direction, we have $F \in Cn(Cn(S)) \Longrightarrow Cn(S) \models F \Longrightarrow^{(*)} S \models F \Longrightarrow F \in Cn(S)$.
  We have (*) because $\mathcal{A} \models Cn(S)$ iff $\mathcal{A} \models S$ by definition of $Cn$.

**Homework 11.4. [Elementary Classes]** (++)

In this exercise, we assume that all structures and formulas share the same signature $\Sigma$.

We define the operator $Mod(S)$ that returns the class of all structures that model a set of formulas $S$. In other words, $Mod(S)$ contains all $\mathcal{A}$ such that $\mathcal{A} \models S$.

A class of models $M$ is said to be *elementary* if there is a set of formulas $S$ such that $M = Mod(S)$. If $S$ is just a singleton set, i.e. there is a formula $F$ such that $S = \{F\}$, then $M$ is *basic elementary.*

Prove:

1. A class of models $M$ is basic elementary if and only if there is a *finite* set of formulas $S$ such that $M = Mod(S)$.
2. If $M$ is basic elementary and $M = Mod(S)$ for countable $S$, then there is a finite subset $S' \subseteq S$ such that $M = Mod(S')$.

**Solution:**

For the first task, simply take $F \coloneqq \bigwedge_{G \in S} G$.

For the second task, it suffices to show that $Mod(S) = Mod(S') \iff Cn(S) = Cn(S')$. The result then follows from tutorial exercise 11.3. Here's the direction from left to right:

$$\begin{aligned} F \in Cn(S) &\iff \mathcal{M} \models F \text{ for any model } \mathcal{M} \text{ of } S \\ &\iff \mathcal{M} \models F \text{ for any } \mathcal{M} \in Mod(S) \\ &\iff \mathcal{M} \models F \text{ for any } \mathcal{M} \in Mod(S') \\ &\iff \mathcal{M} \models F \text{ for any model } \mathcal{M} \text{ of } S' \\ &\iff F \in Cn(S') \end{aligned}$$

The other direction is similar.

The logic of the world is prior to all truth and falsehood.

— Ludwig Wittgenstein[1]

[1] Yes, Ludwig strikes again – he just dropped too many great quotes.