Logic Exercises

Technical University of Munich
Chair for Logic and Verification

Prof. Tobias Nipkow
Kevin Kappelmann

SS 2022 — Exercise Sheet 12 — 15.07.2022

Exercise 12.1. **[Łoś–Vaught Test]**
Given a theory T, one often wants to know whether T is complete, i.e. T contains either F or $\neg F$ for any sentence F. In the lecture, you proved that a theory T is complete iff all its models are elementarily equivalent. However, checking whether all models of a theory are elementarily equivalent is usually rather difficult. The Łoś–Vaught test provides an improved version of this theorem:

Let T be a Σ-theory with no finite models. Let $\kappa \geq |\Sigma|$ be a cardinal. Show that if all models of size κ for T are elementarily equivalent, then T is complete.

You can assume the following without a proof:

Theorem 1 (Generalised Löwenheim-Skolem Theorems)**.** *Let S be a set of formulas in a language of cardinality λ, and assume that S has some infinite model. Then for every infinite cardinal $\kappa \geq \lambda$, there is a model of cardinality κ for S.*

Exercise 12.2. **[QE for DLO]**
Use the quantifier-elimination procedure for DLOs to check whether the following formula is a member of $Th(DLO)$:

$$\exists x \forall y \exists z ((x < y \lor z < x) \land y < z)$$

Use $\iff$ if two formulas are logically equivalent and $\iff_{DLO}$ if the equivalence requires the DLO axioms.

Exercise 12.3. **[Fourier–Motzkin Elimination]**
Apply the Fourier–Motzkin Elimination to check the following sentences:

1. $\exists x \exists y (2 \cdot x + 3 \cdot y = 7 \land x < y \land 0 < x)$
2. $\exists x \exists y (3 \cdot x + 3 \cdot y < 8 \land 8 < 3 \cdot x + 2 \cdot y)$

Use $\iff$ if two formulas are logically equivalent and $\iff_{R_+}$ if the equivalence requires the theory R_+.

Homework 12.1. **[Subtraction Logic]** (+++)
We consider a fragment of linear arithmetic, in which atomic formulas only take the form $x - y \leq c$ for variables x and y, and $c \in \mathbb{R}$.

For a finite set S of such difference constraints, we can define a corresponding inequality graph $G(V, E)$, where V is the set of variables of S, and E consists of all the edges (x, y) with weight c for all constraints $x - y \leq c$ of S. Show that the conjuction of all constraints from S is satisfiable iff G does not contain a negative cycle.

How can you use this theorem to obtain a procedure for deciding whether a formula is a member of this fragment where all variables and constants are of the domain $\mathbb{Z}$?

Homework 12.2. **[Min, Max, Abs]** (++)

1. Show that $\mathrm{Th}(\mathbb{R}, 0, 1, <, =, +, \min, \max)$ is decidable, where min and max return the minimum and maximum of two values.
2. Show that $\mathrm{Th}(\mathbb{R}, 0, 1, <, =, +, \min, \max, |\cdot|)$ is decidable, where $|\cdot|$ is the absolute value.

Homework 12.3. **[Optimising DLO]** (++)
DLO suffers from a heavy performance loss because after each step, a DNF needs to be reconstructed. We want to study an optimisation that may avoid this under some circumstances.

Assume that we want to eliminate an $\exists x F$ where

- F contains no negations and quantifiers,
- F contains no $\perp$, and
- all bounds in F are lower bounds for x or all bounds in F are upper bounds for x.

Then, $\exists x F \equiv \top$. Prove the correctness of this optimisation.

> In order to attain the impossible, one must attempt the absurd.
> — Miguel de Cervantes