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Exercise 12.1. **[Łoś–Vaught Test]**
Given a theory T, one often wants to know whether T is complete, i.e. T contains either F or $\neg F$ for any sentence F. In the lecture, you proved that a theory T is complete iff all its models are elementarily equivalent. However, checking whether all models of a theory are elementarily equivalent is usually rather difficult. The Łoś–Vaught test provides an improved version of this theorem:

Let T be a Σ-theory with no finite models. Let $\kappa \geq |\Sigma|$ be a cardinal. Show that if all models of size κ for T are elementarily equivalent, then T is complete.

You can assume the following without a proof:

Theorem 1 (Generalised Löwenheim-Skolem Theorems)**.** *Let S be a set of formulas in a language of cardinality λ, and assume that S has some infinite model. Then for every infinite cardinal $\kappa \geq \lambda$, there is a model of cardinality κ for S.*

Solution:
Prove by contraposition. Assume T is not complete. Hence there is a sentence F such that $T \not\models F$ and $T \not\models \neg F$. Thus $T \cup \{F\}$ and $T \cup \{\neg F\}$ are both satisfiable. Hence there are $\mathcal{M} \models T \cup \{F\}$ and $\mathcal{M}' \models T \cup \{\neg F\}$. As both are models of T, we know that both models are infinite by assumption.

Now by Löwenheim-Skolem, there are $\mathcal{M}_\kappa \models T \cup \{F\}$ and $\mathcal{M}'_\kappa \models T \cup \{\neg F\}$ of cardinality κ. Thus, not all models of size κ of T are elementarily equivalent.

Exercise 12.2. [QE for DLO]
Use the quantifier-elimination procedure for DLOs to check whether the following formula is a member of $Th(DLO)$:

$$\exists x \forall y \exists z((x < y \vee z < x) \wedge y < z)$$

Use $\iff$ if two formulas are logically equivalent and $\iff_{DLO}$ if the equivalence requires the DLO axioms.

Solution:

$$\begin{aligned}
& \exists x \forall y \exists z((x < y \vee z < x) \wedge y < z) \\
\iff \quad & \exists x \forall y \exists z((x < y \wedge y < z) \vee (z < x \wedge y < z)) \\
\iff \quad & \exists x \forall y (\exists z(x < y \wedge y < z) \vee \exists z(z < x \wedge y < z)) \\
\iff \quad & \exists x \forall y ((x < y \wedge \exists z(y < z)) \vee \exists z(z < x \wedge y < z)) \\
\iff_{DLO} \quad & \exists x \forall y ((x < y \wedge \top) \vee \exists z(z < x \wedge y < z)) \\
\iff_{DLO} \quad & \exists x \forall y (x < y \vee y < x) \\
\iff \quad & \exists x \neg \exists y \neg (x < y \vee y < x) \\
\iff_{DLO} \quad & \exists x \neg \exists y ((y < x \vee x = y) \wedge (x < y \vee x = y)) \\
\iff \quad & \exists x \neg \exists y ((y < x \wedge x < y) \vee (y < x \wedge x = y) \vee (x = y \wedge x < y) \vee (x = y)) \\
\iff \quad & \exists x \neg (\exists y(y < x \wedge x < y) \vee \exists y(y < x \wedge x = y) \vee \exists y(x = y \wedge x < y) \vee \exists y(x = y)) \\
\iff_{DLO} \quad & \exists x \neg (x < x \vee x < x \vee x < x \vee \top) \\
\iff_{DLO} \quad & \exists x ((x = x \vee x < x) \wedge (x = x \vee x < x) \wedge (x = x \vee x < x) \wedge \bot) \\
\iff \quad & \exists x ((x = x \wedge \bot) \vee (x = x \wedge x < x \wedge \bot) \vee (x < x \wedge \bot)) \\
\iff \quad & (\exists x(x = x) \wedge \bot) \vee (\exists x(x = x \wedge x < x) \wedge \bot) \vee (\exists x(x < x) \wedge \bot) \\
\iff_{DLO} \quad & (\top \wedge \bot) \vee (\bot \wedge \bot) \vee (\bot \wedge \bot) \\
\iff \quad & \bot \qquad \text{(optional step; not part of QEP)}
\end{aligned}$$

Exercise 12.3. [Fourier–Motzkin Elimination]
Apply the Fourier–Motzkin Elimination to check the following sentences:

1. $\exists x \exists y(2 \cdot x + 3 \cdot y = 7 \land x < y \land 0 < x)$
2. $\exists x \exists y(3 \cdot x + 3 \cdot y < 8 \land 8 < 3 \cdot x + 2 \cdot y)$

Use $\Longleftrightarrow$ if two formulas are logically equivalent and $\Longleftrightarrow_{R_+}$ if the equivalence requires the theory R_+.

Solution:

$$\begin{aligned}
& \exists x \exists y(2 \cdot x + 3 \cdot y = 7 \land x < y \land 0 < x) \\
\Longleftrightarrow \quad & \exists x\, (\exists y\, (2 \cdot x + 3 \cdot y = 7 \land x < y) \land 0 < x) \\
\Longleftrightarrow_{R_+} \quad & \exists x \left(\exists y \left(y = \frac{7}{3} - \frac{2}{3} \cdot x \land x < y \right) \land 0 < x \right) \\
\Longleftrightarrow_{R_+} \quad & \exists x \left(x < \frac{7}{3} - \frac{2}{3} \cdot x \land 0 < x \right) \\
\Longleftrightarrow_{R_+} \quad & \exists x \left(x < \frac{7}{5} \land 0 < x \right) \\
\Longleftrightarrow_{R_+} \quad & 0 < \frac{7}{5} \\
\Longleftrightarrow_{R_+} \quad & \top \qquad \text{(optional step; not part of QEP)}
\end{aligned}$$

$$\begin{aligned}
& \exists x \exists y(3 \cdot x + 3 \cdot y < 8 \land 8 < 3 \cdot x + 2 \cdot y) \\
\Longleftrightarrow_{R_+} \quad & \exists x \exists y \left(y < \frac{8}{3} - x \land 4 - \frac{3}{2} \cdot x < y \right) \\
\Longleftrightarrow_{R_+} \quad & \exists x \left(4 - \frac{3}{2} \cdot x < \frac{8}{3} - x \right) \\
\Longleftrightarrow_{R_+} \quad & \exists x \left(\frac{8}{3} < x \right) \\
\Longleftrightarrow_{R_+} \quad & \top
\end{aligned}$$

Homework 12.1. **[Subtraction Logic]** (+++)
We consider a fragment of linear arithmetic, in which atomic formulas only take the form $x - y \leq c$ for variables x and y, and $c \in \mathbb{R}$.

For a finite set S of such difference constraints, we can define a corresponding inequality graph $G(V, E)$, where V is the set of variables of S, and E consists of all the edges (x, y) with weight c for all constraints $x - y \leq c$ of S. Show that the conjuction of all constraints from S is satisfiable iff G does not contain a negative cycle.

How can you use this theorem to obtain a procedure for deciding whether a formula is a member of this fragment where all variables and constants are of the domain $\mathbb{Z}$?

Solution:
First part: see here, slide 4.

Second part: We first replace any $x = y$ by $x - y \leq 0 \wedge y - x \leq 0$. We can replace any $\neg(x - y \leq 0)$ by $x - y > 0 \equiv y - x < 0 \equiv y - x \leq -1$. Note that the final step is only possible in $\mathbb{Z}$. For $\mathbb{R}$, one would instead have to symbolically compute with a "sufficiently small" δ instead of -1. We can then use the Bellman-Ford algorithm to detect negative cycles.

Homework 12.2. **[Min, Max, Abs]** (++)

1. Show that $\mathrm{Th}(\mathbb{R}, 0, 1, <, =, +, \min, \max)$ is decidable, where min and max return the minimum and maximum of two values.
2. Show that $\mathrm{Th}(\mathbb{R}, 0, 1, <, =, +, \min, \max, |\cdot|)$ is decidable, where $|\cdot|$ is the absolute value.

Solution:

1. Extend Fourier-Motzkin by new steps before applying $qe1ca$ to $\exists x(A_1 \wedge \cdots \wedge A_n) \equiv: \exists x F$:
   (a) If there is some term $\min(t_1, t_2)$ in F, then replace the formula by
   $$\exists x((t_1 < t_2 \rightarrow F[t_1/\min(t_1, t_2)]) \wedge (t_2 < t_1 \vee t_2 = t_1 \rightarrow F[t_2/\min(t_1, t_2)]))$$
   where by abuse of notation, $F[t_1/\min(t_1, t_2)]$ is the formula obtained by replacing all occurences of $\min(t_1, t_2)$ by t_1. Then renormalise the formula and repeat.
   (b) If there is some term $\max(t_1, t_2)$ in F, then replace the formula by
   $$\exists x((t_1 < t_2 \rightarrow F[t_2/\max(t_1, t_2)]) \wedge (t_2 < t_1 \vee t_2 = t_1 \rightarrow F[t_1/\max(t_1, t_2)]))$$
   Then renormalise the formula and repeat.

   As a result, we reduced the theory to the theory of linear real arithmetic, which is decidable.

1. Similar to the previous exercise with an additional step: If there is some term $c \cdot |t|$ in F, then replace the formula by
   $$\exists x((0 < t \vee 0 = t \rightarrow F[t/|t|]) \vee (t < 0 \rightarrow F[(-c) \cdot t/c \cdot |t|]))$$
   Then renormalise the formula and repeat.

Homework 12.3. [Optimising DLO] (++)
DLO suffers from a heavy performance loss because after each step, a DNF needs to be reconstructed. We want to study an optimisation that may avoid this under some circumstances.

Assume that we want to eliminate an $\exists x F$ where

- F contains no negations and quantifiers,
- F contains no $\bot$, and
- all bounds in F are lower bounds for x or all bounds in F are upper bounds for x.

Then, $\exists x F \equiv \top$. Prove the correctness of this optimisation.

Solution:
WLOG assume that F only contains upper bounds (the other case is analagous). Let $\vec{y}$ be the free variables of $\exists F$. We proof by induction on F that there is a witness w for any instantiation $F[\vec{u}/\vec{y}]$ such that $F[\vec{u}/\vec{y}][t/x] \equiv \top$ for any $t \leq w$.

Case $\top$: any w does the job.

Case $x < z$: For any instantiation u of z, we can obtain by the axioms of DLO some t such that $t < z$. We set $w \coloneqq t$.

Case $F_1 \lor F_2$: then by induction $F_1[\vec{u}/\vec{y}][t/x] \equiv \top$ for some w_1 and all $t \leq w_1$ and hence $F_1[\vec{u}/\vec{y}][t/x] \lor F_2[\vec{u}/\vec{y}][t/x] \equiv \top \lor F_2[\vec{u}/\vec{y}][t/x] \equiv \top$

Case $F_1 \land F_2$: then by induction $F_i[\vec{u}/\vec{y}][t_i/x] \equiv \top$ for some w_i and all $t_i \leq w_i$. Set $w \coloneqq w_1$ if $w_1 < w_2$ and $w \coloneqq w_2$ otherwise. Then $F_1[\vec{u}/\vec{y}][t/x] \land F_2[\vec{u}/\vec{y}][t/x] \equiv \top \land \top \equiv \top$ for all $t \leq w$.

all other cases: excluded by assumption.

In order to attain the impossible, one must attempt the absurd.
— Miguel de Cervantes