First-order Predicate Logic
Theories
Definitions

Definition
A **signature** $\Sigma$ is a set of predicate and function symbols.
A **$\Sigma$-formula** is a formula that contains only predicate and function symbols from $\Sigma$.
A **$\Sigma$-structure** is a structure that interprets all predicate and function symbols from $\Sigma$.

Definition
A **sentence** is a closed formula.
In the sequel, $S$ is a set of sentences.
Theories

Definition
A theory is a set of sentences $S$ such that $S$ is closed under consequence: If $S \models F$ and $F$ is closed, then $F \in S$.

Let $\mathcal{A}$ be a $\Sigma$-structure:
$Th(\mathcal{A})$ is the set of all sentences true in $\mathcal{A}$:
$Th(\mathcal{A}) = \{ F \mid F \text{ $\Sigma$-sentence and } \mathcal{A} \models F \}$

Lemma
Let $\mathcal{A}$ be a $\Sigma$-structure and $F$ a $\Sigma$-sentence.
Then $\mathcal{A} \models F$ iff $Th(\mathcal{A}) \models F$.

Corollary
$Th(\mathcal{A})$ is a theory.
Lemma

Let $\mathcal{A}$ be a $\Sigma$-structure and $F$ a $\Sigma$-sentence. Then $\mathcal{A} \models F$ iff $Th(\mathcal{A}) \models F$.

Proof

“$\Rightarrow$”: $\mathcal{A} \models F \Rightarrow F \in Th(\mathcal{A}) \Rightarrow Th(\mathcal{A}) \models F$

“$\Leftarrow$”:
Assume $Th(\mathcal{A}) \models F$
$\Rightarrow$ for all $\mathcal{B}$, if $\mathcal{B} \models Th(\mathcal{A})$ then $\mathcal{B} \models F$
$\Rightarrow \mathcal{A} \models F$ because $\mathcal{A} \models Th(\mathcal{A})$
Example

**Notation:** \( (\mathbb{Z}, +, \leq) \) denotes the structure with universe \( \mathbb{Z} \) and the standard interpretations for the symbols \(+\) and \(\leq\). The same notation is used for other standard structures where the interpretation of a symbol is clear from the symbol.

**Example (Linear integer arithmetic)**

\( Th(\mathbb{Z}, +, \leq) \) is the set of all sentences over the signature \(\{+, \leq\} \) that are true in the structure \((\mathbb{Z}, +, \leq)\).
Famous numerical theories

\( Th(\mathbb{R}, +, \leq) \) is called linear real arithmetic.
It is decidable.

\( Th(\mathbb{R}, +, \ast, \leq) \) is called real arithmetic.
It is decidable.

\( Th(\mathbb{Z}, +, \leq) \) is called linear integer arithmetic or Presburger arithmetic.
It is decidable.

\( Th(\mathbb{Z}, +, \ast, \leq) \) is called integer arithmetic.
It is not even semidecidable (= r.e.).

Decidability via special algorithms.
Consequences

Definition
Let $S$ be a set of $\Sigma$-sentences.

$Cn(S)$ is the set of consequences of $S$:
$Cn(S) = \{ F \mid F \text{ $\Sigma$-sentence and } S \models F \}$

Examples
$Cn(\emptyset)$ is the set of valid sentences.
$Cn(\{\forall x \forall y \forall z (x \ast y) \ast z = x \ast (y \ast z)\})$ is the set of sentences that are true in all semigroups.

Lemma
If $S$ is a set of $\Sigma$-sentences, $Cn(S)$ is a theory.

Proof
Assume $F$ is closed and $Cn(S) \models F$. Show $F \in Cn(S)$, i.e. $S \models F$. Assume $\mathcal{A} \models S$. Thus $\mathcal{A} \models Cn(S)$ (*) and hence $\mathcal{A} \models F$, i.e. $S \models F$. (*): Assume $G \in Cn(S)$, i.e. $S \models G$. With $\mathcal{A} \models S$ the desired $\mathcal{A} \models G$ follows.
Axioms

Definition
Let $S$ be a set of $\Sigma$-sentences.

A theory $T$ is **axiomatized** by $S$ if $T = Cn(S)$

A theory $T$ is **axiomatizable** if there is some decidable or recursively enumerable $S$ that axiomatizes $T$.

A theory $T$ is **finitely axiomatizable** if there is some finite $S$ that axiomatizes $T$. 
Completeness and elementary equivalence

Definition
A theory $T$ is complete if for every sentence $F$, $T \models F$ or $T \models \neg F$.

Fact
$Th(A)$ is complete.

Example
$Cn(\{\forall x \forall y \forall z \ (x \ast y) \ast z = x \ast (y \ast z)\})$ is incomplete: neither $\forall x \forall y \ x \ast y = y \ast x$ nor its negation are present.

Definition
Two structures $\mathcal{A}$ and $\mathcal{B}$ are elementarily equivalent if $Th(\mathcal{A}) = Th(\mathcal{B})$.

Theorem
A theory $T$ is complete iff all its models are elementarily equivalent.
**Theorem**

A theory $T$ is complete iff all its models are elementarily equivalent.

**Proof** If $T$ is unsatisfiable, then $T$ is complete (because $T \models F$ for all $F$) and all models are elementarily equivalent.

Now assume $T$ has a model $M$.

“⇒”

Assume $T$ is complete. Let $F \in Th(M)$. We cannot have $T \models \neg F$ because $M \models T$ would imply $M \models \neg F$ but $M \models F$ because $F \in Th(M)$. Thus $T \models F$ by completeness. Therefore every formula that is true in some model of $T$ is true in all models of $T$.

“⇐”

Assume all models of $T$ are elem.eq. Let $F$ be closed. Either $M \models F$ or $M \models \neg F$. By elem.eq. $T \models F$ or $T \models \neg F$.

Why? Assume $M \models F$ (similar for $M \models \neg F$).

To show $T \models F$, assume $A \models T$ and show $A \models F$.

$\Rightarrow Th(A) = Th(M)$ by elem.eq.

$\Rightarrow$ for all closed $F$, $A \models F$ iff $M \models F$

$\Rightarrow A \models F$ because $M \models F$