Technische Universität München
Institut für Informatik
Prof. Tobias Nipkow, Ph.D.
Brian Huffman and Peter Lammich

Equational Logic
Summer Term 2012
Exercise Sheet 2
April 27

Homework is due on May 4, before the tutorial.

Exercise 1 (H) (*Termination*)

Let $\longrightarrow$ be a relation over the set $\mathbb{N}^*$ of lists of natural numbers, defined as follows.

$$[n+1, n_1, \ldots, n_k] \longrightarrow [n, n_1, \ldots, n_k, n] \quad \text{for any } n, k, n_{1\ldots k} \geq 0$$
$$[0, n_1, \ldots, n_k] \longrightarrow [n_1, \ldots, n_k] \quad \text{for any } k, n_{1\ldots k} \geq 0$$

Prove that $\longrightarrow$ is terminating.

Exercise 2 (H) (*Strict Order and Termination*)

We define the *reverse dictionary ordering* on strings in $\{\mathrm{a}, \mathrm{b}\}^*$ such that $w_1 \longrightarrow w_2$ if and only if w_2 would come before w_1 in a dictionary. (We consider the empty string ε to be the first entry in the dictionary.) For example:

$$\mathrm{b} \longrightarrow \mathrm{ab} \longrightarrow \mathrm{aa} \longrightarrow \mathrm{a} \longrightarrow \varepsilon$$

a) The following formal, inductive definition of $\longrightarrow$ is incomplete. Add any missing rule(s) needed to complete the definition.

  1a. If $w_1 \longrightarrow w_2$, then $\mathrm{a}w_1 \longrightarrow \mathrm{a}w_2$.

  1b. If $w_1 \longrightarrow w_2$, then $\mathrm{b}w_1 \longrightarrow \mathrm{b}w_2$.

  2. $\mathrm{b}w_1 \longrightarrow \mathrm{a}w_2$.

  ...

b) Show that $\longrightarrow$ is a strict partial order, i.e., that it is irreflexive, asymmetric, and transitive. Hint: As $\longrightarrow$ is defined inductively, you can prove such properties by induction, with cases for each of the rules above.

c) Show that $\longrightarrow$ is non-terminating, by demonstrating an infinite reduction sequence.

Exercise 3 (T) (*Termination*)

Show that the following programs, with variables over the natural numbers, terminate on all valid inputs.

a) ggT(m, n)

$$\begin{array}{l} \textbf{while } m \neq n \textbf{ do} \\ \qquad \textbf{if } m > n \textbf{ then } m := m - n \textbf{ else } n := n - m \end{array}$$

b) ggT(m, n)

$$\begin{array}{l} \textbf{while } m \neq n \textbf{ do} \\ \qquad \textbf{if } m > n \textbf{ then } m := m - n \\ \qquad \textbf{else begin } h := m;\ m := n;\ n := h \textbf{ end} \end{array}$$

c) The function f, which is defined recursively as follows.

$$\begin{array}{rcl} f(m, n, 0) & = & m + n \\ f(m, 0, k + 1) & = & f(m + k, 1, k) \\ f(m, n + 1, k + 1) & = & f(m + k, f(m + k, n, k + 1), k) \end{array}$$

Exercise 4 (T) (*Product Order*)

Show that the lexicographic product $(A \times B, >_{A \times B})$ of the orders $(A, >_A)$ and $(B, >_B)$ is a total order, if $>_A$ and $>_B$ are total orders.

Exercise 5 (T) (*Measure Function*)

Let $(\{0, 1\}^*, \longrightarrow)$ be the reduction system over strings from the alphabet $\{0, 1\}$ with the single reduction rule $u1v0w \longrightarrow u0v1w$, where $u, v, w \in \{0, 1\}^*$.

Give a measure function $\varphi : \{0, 1\}^* \to \mathbb{N}$ showing that the reduction system terminates.