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Equational Logic
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Solutions to Exercise Sheet 1

Every sheet contains *exercises* and *homework* assignments. We strongly recommend you prepare for the exercise sessions by reading the exercises on the sheet and make yourself familiar with the concepts. Homework assignments are due the following week after the sheet was published, to be handed in before the exercise session. You have to do the homework assignments yourself. Team work is not allowed!

Exercise 1 (Warm-Up)

Which of the following closure operators commute? Prove or refute!

a) $\longleftrightarrow^+ = \xrightarrow{+} \cup (\xrightarrow{+})^{-1}$

b) $\longleftarrow^+ = (\xrightarrow{+})^{-1}$

Solution

a) Counterexample: $a_1 \longrightarrow a_2$

$$\begin{aligned} \longleftrightarrow^+ &= \{(a_2, a_1), (a_1, a_2), (a_1, a_1), (a_2, a_2)\} \\ \xrightarrow{+} \cup (\xrightarrow{+})^{-1} &= \{(a_1, a_2), (a_2, a_1)\} \end{aligned}$$

b) **"$\Longrightarrow$":** To show: $a \xleftarrow{+} b \implies b \xrightarrow{+} a$. Obvious.

"$\Longleftarrow$": To show: $b \xrightarrow{+} a \implies a \xleftarrow{+} b$. Obvious.

Exercise 2 (Bounded Relations)

A relation $\longrightarrow$ over the set A is called *bounded*, if for each element x, the lengths of all paths from x are bounded. Formally:

$$\forall x \in A.\ \exists n.\ \nexists y \in A.\ x \xrightarrow{n} y$$

Prove or refute:

a) Each terminating relation is bounded.

b) A finitely branching relation is terminating if and only if it is bounded. (Hint: Well-founded induction)

c) Now we call a relation *globally bounded*, if there is a bound that is valid for all elements. Formally:

$$\exists n.\ \forall x \in A.\ \nexists y \in A.\ x \xrightarrow{n} y$$

Prove or refute: Any finitely branching and terminating relation is globally bounded.

Solution

a) Not every terminating relation is bounded. Example: the relation R over the set $\mathbb{N} \cup \{\bot\}$

$$R = \{(\bot, n) \mid n \in \mathbb{N}\} \cup \{(n+1, n) \mid n \in \mathbb{N}\}$$

$$\begin{tikzcd}
0 & 1 \arrow[l] & 2 \arrow[l] & 3 \arrow[l] & \dots \arrow[l] \\
& & \bot \arrow[ull] \arrow[ul] \arrow[u] \arrow[ur] \arrow[urr] & &
\end{tikzcd}$$

R is terminating: For all $x \in \mathbb{N} \cup \{\bot\}$, eventually $x \xrightarrow{*} 0$
R is unbounded: There is no upper bound on reduction sequences starting at $\bot$.
Paths of all non-zero lengths exist, $\forall n > 0.\ \bot \xrightarrow{n} 0$.

b) A finitely branching relation is terminating if and only if it is bounded.
Bounded $\Rightarrow$ terminating: Because there is a bound on the lengths of all reduction sequences, it follows that there can be no infinite reduction sequence.
Terminating $\Rightarrow$ bounded: Proof by well-founded induction.

We again state the induction principle:

$$\text{Rule: } \frac{\forall x \in A.\ \overbrace{(\forall y \in A.\ x \xrightarrow{+} y \Rightarrow P(y))}^{\text{Induction hypothesis}} \Rightarrow P(x)}{\forall x \in A.\ P(x)}$$

Proof. Show $\forall x \in A.\ \underbrace{\exists n.\ \nexists y \in A.\ x \xrightarrow{n} y}_{P(x)}$.

With well-founded induction, we get the induction hypothesis, that $P(y)$ holds for all successors y of x (i.e., $x \xrightarrow{m} y$).

We show $P(x)$ by a case distinction.

x **is in normal form:** Then x has no successors, so $P(x)$ holds with $n = 1$.

x **is not in normal form:** Then x has finitely many immediate successors., i.e., $x \longrightarrow y$ for finitely many y. From the induction hypothesis, we know that for each immediate successor y of x there is a bound n_y, such that the sucessors of y are reachable by at most n_y steps from y.

In particular, this holds for the direct successors of x. By assumption, the relation is finitely branching, i.e., x has only finitely many sucessors, such that the maximum

$$n_{\max} = \max\{n_y | x \longrightarrow y\}$$

is defined. Thus, we choose $n = n_{\max} + 1$.

$\square$

c) Not every finitely-branching and terminating relation is globally bounded. Example: the relation R over the set $\mathbb{N}$

$$R = \{(n+1, n) \mid n \in \mathbb{N}\}$$

As a sequence:

$$0 \leftarrow 1 \leftarrow 2 \leftarrow 3 \leftarrow \dots$$

The relation R is finitely-branching and bounded (for each i we get in i steps to 0) hence terminating. But not globally-bounded, as for each bound i the element $i+1$ needs $i+1$ steps to terminate.

Exercise 3 (Partial Ordering)

Prove or refute:

a) $\xrightarrow{+}$ is a strict partial order if and only if $\longrightarrow$ is acyclic.

b) $\xrightarrow{*}$ is a partial order if and only if $\longrightarrow$ is acyclic.

Notes: A relation $R \subseteq X \times X$ is called *strict partial order* if it is irreflexive ($\forall x \in X.\ \neg(x\ R\ x)$), transitive ($\forall x, y, z \in X.\ x\ R\ y \wedge y\ R\ z \implies x\ R\ z$), and asymmetric ($\forall x, y \in X.\ x\ R\ y \implies \neg(y\ R\ x)$).

A relation $R \subseteq X \times X$ is called *partial order* if it is reflexive ($\forall x \in X.\ x\ R\ x$), transitive and antisymmetric ($\forall x, y \in X.\ x\ R\ y \wedge y\ R\ x \implies x = y$).

A relation $\longrightarrow \subseteq X \times X$ is called *acyclic* if there is no element a, s.t. $a \xrightarrow{+} a$.

Solution

a) $\xrightarrow{+}$ is a strict partial order if and only if $\longrightarrow$ is acyclic.

Proof. From irreflexivity of $\xrightarrow{+}$ follows $\longrightarrow$ is acyclic.

By definition $\xrightarrow{+}$ is transitive. As $\longrightarrow$ is acyclic, $\xrightarrow{+}$ is irreflexive. Would be $\xrightarrow{+}$ not asymmetric, i.e. $a \xrightarrow{+} b$ and $b \xrightarrow{+} a$ then by transitivity $a \xrightarrow{+} a$, which contradicts irreflexivity.

□

b) Counterexample: For $R = \{(a, a) \mid a \in \mathbb{N}\}$, the reflexive and transitive closure R^* is a partial order. But R is not acyclic!

Exercise 4 (Example)

Let $(M, \longrightarrow)$ be a reduction system with $M = \{A_1, A_2, A_3, A_4, B_1, B_2, B_3, C_1, C_2, C_3, C_4, D, E\}$ and $\longrightarrow$ defined as follows:

- $A_1 \longrightarrow B_1$, $A_1 \longrightarrow B_2$, $A_2 \longrightarrow B_1$, $A_2 \longrightarrow B_2$, $A_3 \longrightarrow B_3$, $A_4 \longrightarrow B_3$,
- $B_1 \longrightarrow C_1$, $B_2 \longrightarrow C_2$, $B_2 \longrightarrow C_3$, $B_3 \longrightarrow C_1$, $B_3 \longrightarrow C_2$, $B_3 \longrightarrow C_3$, $B_3 \longrightarrow C_4$,
- $C_3 \longrightarrow E$, $C_4 \longrightarrow E$, and $D \longrightarrow C_4$.

Which of the following properties are satisfied by $\longrightarrow$? Give a justification.
terminating, globally bounded, asymmetric, antisymmetric, reflexive, irreflexive, transitive

Solution

$\longrightarrow$ is obviously finitely branching and terminating, hence globally finite. It is not reflexive (counterexample: E). It is irreflexive and asymmetric. It is trivially antisymmetric, because there are no symmetric pairs in $\longrightarrow$. It is not transitive (counterexample: $A_1 \longrightarrow B_1 \longrightarrow C_1$, but not $A_1 \longrightarrow C_1$.

Homework 5 (Primes)

Let $(\mathbb{N}_{>0}, \longrightarrow)$ be the reduction system on positive natural numbers, where

$$\longrightarrow = \{(n, m) \mid 11n = 2m \vee 5n = 13m\}$$

a) Does this system terminate? Justify your answer.

b) Determine the set of all irreducible elements.

c) What is the normal form of 1210? Show: $10 \longleftrightarrow 26$ and $10 \overset{*}{\longleftrightarrow} 143$.

Homework 6 (Equivalence Relation)

A relation $R \subseteq X \times X$ is called an *equivalence relation*, if:

- R is reflexive, i.e. $\forall x \in X.\ x\ R\ x$
- R is transitive, i.e. $\forall x, y, z \in X.\ x\ R\ y \wedge y\ R\ z \implies x\ R\ z$
- R is symmetric, i.e. $\forall x, y \in X.\ x\ R\ y \implies y\ R\ x$

Let $\longrightarrow$ be a relation. Show: $\overset{*}{\longleftrightarrow}$ is the smallest equivalence relation that contains $\longrightarrow$.

Homework 7 (Confluence And Normal Form)

a) Show that a reduction system $(A, \longrightarrow)$ is confluent and normalizing, if and only if every element has a unique normal form.

b) Give an example of a confluent and normalizing reduction system that does not terminate.