Logics Exercise

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SS 2016 | Exercise Sheet 8 | 01.06.2016

Submission of Homework: Before tutorial on June 8

Exercise 8.1. **[Garnix and Fourier-Motzkin]**
A university brewery has two kegging machines (M_1 and M_2) and produces two products, beer (B) and shandy (R). M_1 needs 50 minutes to fill a keg of beer but only 24 minutes to fill a keg of shandy, while M_2 needs 30 minutes for a keg of either one drink.

For a student festival beginning two weeks from now, an order of 75 kegs of beer and 95 kegs of shandy has been placed, which has to be fulfilled during the upcoming week. However, machine M_1 has to undergo maintance and therefore is expected to be running between 37 and 42 hours, while M_2 can be operated up to 100 hours during the whole week (and as few hours as desirable).

Use Fourier-Motzkin elimination to check wether the brewery can hold up to its promise.

Exercise 8.2. **[Refining Fourier-Motzkin]**
Show how Fourier-Motzkin elimination can be extended to directly handle constraints of the form $x \leq y$ instead of rewriting them to $x < y \vee x = y$ first.

Exercise 8.3. **[Difference Logic]**
We consider a fragment of linear arithmetic, in which atomic formulas only take the form $x - y \leq c$ for variables x and y, and $c \in \mathbb{Q}$.

For a finite set S of such difference constraints, we can define a corresponding inequality graph $G(V, E)$, where V is the set of variables of S, and E consists of all the edges (x, y) with weight c for all constraints $x - y \leq c$ of S. Show that the conjuction of all constraints from a S is satisfiable iff G does not contain a negative cycle.

How can you use this theorem to obtain a procedure for deciding wether a formula is a member of this fragment?

Homework 8.1. [Loop Parallelization] (5 points)
Consider a program of the following form:

```
for (i=l1; i<u1; i++)
  for (j=l2; j<u2; j++)
    A[x1*i+x2*j+o1] = A[y1*i+y2*j+o2]
```

a) Give a formula in linear integer arithmetic that specifies whether the loop can be parallelized, i.e., whether

- No two iterations write to the same array index
- No iteration writes to an array index which is read by another iteration

b) Use quantifier elimination to decide the following formula over $\mathcal{Z}_+$

$$\neg\exists i \exists j (3i \leq 5j \wedge 5j \leq 3i \wedge 1 \leq i)$$

You are allowed to perform simplifications after the steps, e.g., to rewrite literals of the form $cx + k \leq cx$ for $k > 0$ to $\perp$.

Note: This formula states whether the following loop can be parallelized, where ? denotes some unknown value, e.g., depending on input:

```
for (i=1;i<?;i++) A[3*i] = A[5*i]
```

Homework 8.2. [Quantifier Elimination for intervals of $\mathbb{R}$] (5 points)
Find a quantifier-elimination procedure for

$$Th(\{x \in \mathbb{R} | 0 \leq x \leq 1\}, <, =)$$

Can you also find a quantifier-elimination procedure for the following theory?

$$Th(\{x \in \mathbb{R} | 0 < x < 1\}, <, =)$$

Homework 8.3. [Min, Max, Abs] (5 points)
a) Show that $Th(\mathbb{R}, 0, 1, <, =, +, \min, \max)$ is decidable, where min/max return the minimum/ maximum of two values.

b) Show that $Th(\mathbb{R}, 0, 1, <, =, +, \min, \max, |\cdot|)$ is decidable, where $|\cdot|$ is the absolute value.

Hint: Reduce the problems to $Th(\mathbb{R}, 0, 1, <, =, +)$.

Homework 8.4. [Ferrante-Rackoff] (5 points)
Use the Ferrante-Rackoff procedure to decide the example from Exercise 1.