Logics Exercise

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Exercise 4.1. [**(In)finite Models**]

1. Show that any model (for a formula of predicate logic) with an universe of size n can be extended to a model of size m for any $m \geq n$. Can it also be extended to an *infinite* model?
2. Now consider the extension of predicate logic with equality. Does above property still hold?

Exercise 4.2. [**Decidability and Context-Free Grammars**]

Give an alternative proof that is impossible to decide validity of predicate logic formulas by using an encoding of context-free grammars in predicate logic.

Hint: Consider Chomsky normal forms. It is impossible to decide if two context-free languages are disjoint.

Homework 4.1. [Decidability of Consequence] (5 points)
Given a finite set M of (predicate logic) formulas, and a formula F. Is it semi-decidable whether $M \models F$? Is it even decidable? Justify your answers!

Solution: It is semi-decidable whether M holds. Let $M = \{F_1, \ldots, F_n\}$. Consider the formula $(\bigwedge_{i=1,..n,} F_n) \rightarrow F$. By induction over n it follows that $M \models F$ iff $\models (\bigwedge_{i=1,..n,} F_n) \rightarrow F$. The latter question is semi-decidable by first negating the formula and then running e.g. Gilmore's algorithm.

However, the question $M \models F$ is not decidable. Consider $M = \emptyset$, then $M \models F$ iff $\models F$, which is undecidable (see lecture). Alternatively, set M to only contain tautologies.

Homework 4.2. [Ground Resolution] (5 points)
Use ground resolution to prove that the following formula is valid:

$$(\forall x P(x, f(x))) \longrightarrow \exists y P(c, y)$$

Solution:

$$\begin{aligned}
&\neg((\forall x P(x, f(x))) \longrightarrow \exists y P(c, y)) \\
&(\forall x P(x, f(x))) \wedge \neg \exists y P(c, y)) \\
&(\forall x P(x, f(x))) \wedge \forall y \neg P(c, y)) \\
&\forall x \forall y (P(x, f(x)) \wedge \neg P(c, y)) && \text{(Skolem-Form)}
\end{aligned}$$

Now enumerate the Herbrand expansion:

$$E(F) = \{P(c, f(c)) \wedge \neg P(c, f(c)), \ldots\}$$

With resolution, we immediately get $\square$ from the first item in the enumeration.

Homework 4.3. [Formulas without Negation] (5 points)
Prove that every predicate logic formula that only contains $\wedge, \vee, \forall, \exists, \longrightarrow$ and atomic formulas is satisfiable. Is such a formula also valid?

Solution: Choose a suitable structure $\mathcal{A}$ that interprets all predicates to be true everywhere. Then, by straightforward induction on the formula, we get that $\mathcal{A}$ is a model.

However, the formula needs not to be valid. Consider, e.g., the formula P for a nullary predicate P. This is clearly not valid, as there are models that interpret P not to hold.

Homework 4.4. [Herbrand Models] (5 points)
Given the formula

$$F = \forall x \forall y (P(f(x), g(y)) \wedge \neg P(g(x), f(y)))$$

a) Specify a Herbrand model for F.
b) Specify a Herbrand structure suitable for F, which is not a model of F.

Solution: We define $U_{\mathcal{A}} = D(F)$, i.e., the Herbrand universe for F. Note that we have a constant $a \in D(F)$. We define $f^{\mathcal{A}}$ and $g^{\mathcal{A}}$ to be the Herbrand-interpretations.

a) We define $P^{\mathcal{A}} = \{(f(t_1), g(t_2)) \mid t_1, t_2 \in D(F)\}$
b) We define $P^{\mathcal{A}} = \{(g(t_1), f(t_2)) \mid t_1, t_2 \in D(F)\}$