Basic Proof Theory
Propositional Logic

(See the book by Troelstra and Schwichtenberg)

Abbreviations

Until further notice:

$\bot$, $\neg$, $\wedge$, $\vee$, $\rightarrow$ are primitives.

$\top$ abbreviates $\neg\bot$

A possible simplification:

$\neg F$ abbreviates $F \rightarrow \bot$

Sequent Calculus rules

$$\frac{}{\bot, \Gamma \Rightarrow \Delta} \bot L \qquad \frac{}{A, \Gamma \Rightarrow A, \Delta} Ax$$

$$\frac{\Gamma \Rightarrow F, \Delta}{\neg F, \Gamma \Rightarrow \Delta} \neg L \qquad \frac{F, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \neg F, \Delta} \neg R$$

$$\frac{F, G, \Gamma \Rightarrow \Delta}{F \wedge G, \Gamma \Rightarrow \Delta} \wedge L \qquad \frac{\Gamma \Rightarrow F, \Delta \quad \Gamma \Rightarrow G, \Delta}{\Gamma \Rightarrow F \wedge G, \Delta} \wedge R$$

$$\frac{F, \Gamma \Rightarrow \Delta \quad G, \Gamma \Rightarrow \Delta}{F \vee G, \Gamma \Rightarrow \Delta} \vee L \qquad \frac{\Gamma \Rightarrow F, G, \Delta}{\Gamma \Rightarrow F \vee G, \Delta} \vee R$$

$$\frac{\Gamma \Rightarrow F, \Delta \quad G, \Gamma \Rightarrow \Delta}{F \rightarrow G, \Gamma \Rightarrow \Delta} \rightarrow L \qquad \frac{F, \Gamma \Rightarrow G, \Delta}{\Gamma \Rightarrow F \rightarrow G, \Delta} \rightarrow R$$

Notation: $\vdash_G S$ means that there is a proof tree for S using the above rules.

Admissible rules

Definition

A rule

$$\frac{S_1 \quad \dots \quad S_n}{S_{n+1}}$$

is admissible if $\vdash_G S_1, \dots, \vdash_G S_n$ together imply $\vdash_G S_{n+1}$.

Notation: $\vdash_n S$ means that there is a proof tree for S with depth $\leq n$.

Admissible rules

Lemma (Non-atomic Ax)

$\vdash_G F, \Gamma \Rightarrow F, \Delta.$

Lemma (Weakening)

If $\vdash_n \Gamma \Rightarrow \Delta$ *then* $\vdash_n \Gamma', \Gamma \Rightarrow \Delta', \Delta.$

Admissible rules

Lemma (Inversion rules)

$\wedge L^{-1}$ *If* $\vdash_n F \wedge G, \Gamma \Rightarrow \Delta$ *then* $\vdash_n F, G, \Gamma \Rightarrow \Delta$

$\vee R^{-1}$ *If* $\vdash_n \Gamma \Rightarrow F \vee G, \Delta$ *then* $\vdash_n \Gamma \Rightarrow F, G, \Delta$

$\wedge R^{-1}$ *If* $\vdash_n \Gamma \Rightarrow F_1 \wedge F_2, \Delta$ *then* $\vdash_n \Gamma \Rightarrow F_i, \Delta$ $(i = 1, 2)$

$\vee L^{-1}$ *If* $\vdash_n F_1 \vee F2, \Gamma \Rightarrow \Delta$ *then* $\vdash_n F_i, \Gamma \Rightarrow \Delta$ $(i = 1, 2)$

$\rightarrow R^{-1}$ *If* $\vdash_n \Gamma \Rightarrow F \rightarrow G, \Delta$ *then* $\vdash_n F, \Gamma \Rightarrow G, \Delta$

$\rightarrow L^{-1}$ *If* $\vdash_n F \rightarrow G, \Gamma \Rightarrow \Delta$
then $\vdash_n \Gamma \Rightarrow F, \Delta$ *and* $\vdash_n G, \Gamma \Rightarrow \Delta$

Admissible rules

Lemma (Contraction)

- *If* $\vdash_n F, F, \Gamma \Rightarrow \Delta$ *then* $\vdash_n F, \Gamma \Rightarrow \Delta$
- *If* $\vdash_n \Gamma \Rightarrow F, F, \Delta$ *then* $\vdash_n \Gamma \Rightarrow F, \Delta$

Lemma

If $\vdash_G \Gamma \Rightarrow \Delta$ *then* $\vdash_G \Gamma \Rightarrow \Delta - \{\bot\}$

Lemma (Atomic Cut)

If $\vdash_G \Gamma \Rightarrow A, \Delta$ *and* $\vdash_G A, \Gamma \Rightarrow \Delta$ *then* $\vdash_G \Gamma \Rightarrow \Delta$

Lemma (Cut)

If $\vdash_G \Gamma \Rightarrow F, \Delta$ *and* $\vdash_G F, \Gamma \Rightarrow \Delta$ *then* $\vdash_G \Gamma \Rightarrow \Delta$

Natural Deduction rules

$$\frac{F \quad G}{F \wedge G}\ \wedge I \qquad\qquad \frac{F \wedge G}{F}\ \wedge E_1 \quad \frac{F \wedge G}{G}\ \wedge E_2$$

$$\frac{\begin{matrix}[F] \\ \vdots \\ G\end{matrix}}{F \rightarrow G}\ \rightarrow I \qquad\qquad \frac{F \rightarrow G \quad F}{G}\ \rightarrow E$$

$$\frac{F}{F \vee G}\ \vee I_1 \quad \frac{G}{F \vee G}\ \vee I_2 \qquad \frac{F \vee G \quad \begin{matrix}[F] \\ \vdots \\ H\end{matrix} \quad \begin{matrix}[G] \\ \vdots \\ H\end{matrix}}{H}\ \vee E$$

$$\frac{\begin{matrix}[\neg F] \\ \vdots \\ \bot\end{matrix}}{F}\ \bot$$

Natural Deduction rules

Rules for $\neg$ are special cases of rules for $\rightarrow$:

$$\frac{\begin{array}{c}[F]\\ \vdots\\ \bot\end{array}}{\neg F}\ \neg I \qquad \frac{\neg F \quad F}{\bot}\ \neg E$$

Hilbert System

The rule $\rightarrow E$ together with the following axioms:

$$
\begin{array}{ll}
F \rightarrow G \rightarrow F & (A_1) \\
(F \rightarrow G \rightarrow H) \rightarrow (F \rightarrow G) \rightarrow F \rightarrow H & (A_2) \\
F \rightarrow G \rightarrow F \wedge G & (A_3) \\
F \wedge G \rightarrow F & (A_4) \\
F \wedge G \rightarrow G & (A_5) \\
F \rightarrow F \vee G & (A_6) \\
G \rightarrow F \vee G & (A_7) \\
F \vee G \rightarrow (F \rightarrow H) \rightarrow (G \rightarrow H) \rightarrow H & (A_8) \\
(\neg F \rightarrow \bot) \rightarrow F & (A_9)
\end{array}
$$

Convention: $\rightarrow$ associates to the right:
$F \rightarrow G \rightarrow H$ means $F \rightarrow (G \rightarrow H)$