TU München
Institut für Informatik

Prof. Tobias Nipkow
Lars Hupel

SS 2017 | Exercise Sheet 5 | 30.05.2017

Submission of homework: Before tutorial on **Wednesday,** 07.06.2017. You have to do the homework yourself; no teamwork allowed.

Exercise 5.1. **[From Natural Deduction to Hilbert Calculus]**
Prove the following formula to a linear proof in Hilbert calculus:

$$(F \wedge G) \rightarrow (G \wedge F)$$

Hint: Use the deduction theorem.

Exercise 5.2. **[Equivalence]**
Let F and G be arbitrary formulas. (In particular, they may contain free occurrences of x.) Which of the following equivalences hold? Proof or counterexample!

1. $\forall x(F \wedge G) \equiv \forall xF \wedge \forall xG$
2. $\exists x(F \wedge G) \equiv \exists xF \wedge \exists xG$

Exercise 5.3. **[Infinite Models]**
Consider predicate logic with equality. We use infix notation for equality, and abbreviate $\neg(s = t)$ by $s \neq t$. Moreover, we call a structure finite iff its universe is finite.

1. Specify a finite model for the formula $\forall x\ (c \neq f(x) \wedge x \neq f(x))$.
2. Specify a model for the formula $\forall x \forall y\ (c \neq f(x) \wedge (f(x) = f(y) \longrightarrow x = y))$.
3. Show that the above formula has no finite model.

Homework 5.1. **[A Smaller Set of Axioms for Hilbert Calculus]** (6 points)
In the lecture, Hilbert calculus for propositional logic was introduced by means of nine axioms. However, the following three axioms are already sufficient:

A1 $F \to (G \to F)$

A2 $(F \to G \to H) \to (F \to G) \to F \to H$

A10 $(\neg F \to \neg G) \to (G \to F)$

Derive the following axioms from the axioms above with help of $\to_E$:

1. $\neg(F \to F) \to G$
2. $\neg\neg F \to F$

Homework 5.2. **[Predicate Logic]** (6 points)

1. Specify a satisfiable formula F such that for all models $\mathcal{A}$ of F, we have $|U_\mathcal{A}| \geq 3$.
2. Can you also specify a satisfiable formula F such that for all models $\mathcal{A}$ of F, we have $|U_\mathcal{A}| \leq 3$?

Homework 5.3. **[Orders]** (8 points)
A reflexive and transitive relation is called *preorder*. In predicate logic, preorders can be characterized by the formula

$$F \equiv \forall x \forall y \forall z \; (P(x,x) \wedge (P(x,y) \wedge P(y,z) \to P(x,z)))$$

Which of the following structures are models of F? Give an informal proof in the positive case and a counterexample for the negative case!

1. $U^\mathcal{A} = \mathbb{N}$ and $P^\mathcal{A} = \{(m,n) \mid m < n\}$
2. $U^\mathcal{A} = \mathbb{Z} \times \mathbb{Z}$ and $P^\mathcal{A} = \{((x,y),(a,b)) \mid \; a - b \leq x - y \;\}$
3. $U^\mathcal{A} = \mathbb{R}$ and $P^\mathcal{A} = \{(m,n) \mid m = n\}$
4. Let $Q(x,y)$ be specified as follows: $\forall x \forall y (P(x,y) \leftrightarrow Q(y,x))$. Is Q a preorder?

Specify the notion of *partial orders*, that is, preorders that additionally satisfy antisymmetry.