Logics Exercise

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Submission of homework: Before tutorial on 04.07.2017. You have to do the homework yourself; no teamwork allowed.

Exercise 9.1. [**Models of the $\exists^*\forall^*$ Class**]
Consider the $\exists^*\forall^*$ class, i.e. formulas of the form

$$\exists x_1 \dots \exists x_n \; \forall y_1 \dots \forall y_m \; F$$

where F is quantifier-free and contains no function symbols. Show that such a formula has a model iff it has a model of size n (assuming $n \geq 1$). What happens if we allow equality in F?

Exercise 9.2. [**Miniscoping**]
In the lecture, we proved that deciding unsatisfiability of monadic FOL formulas can be reduced to deciding unsatisfiability of formulas from the $\exists^*\forall^*$ fragment by using miniscoping.

Prove the lemma that after miniscoping, no nested quantifiers remain.

Exercise 9.3. [$\exists^*\forall^*$ **with Equality**]
Show that unsatisfiability of formulas from the $\exists^*\forall^*$ fragment with equality is decidable.

Hint: Reduce it to the $\exists^*\forall^*$-fragment without equality.

Exercise 9.4. [$\exists^*\forall^2\exists^*$]
Show how to reduce deciding unsatisfiability of formulas from the $\exists^*\forall^2\exists^*$-fragment to deciding unsatisfiability of formulas from the $\forall^2\exists^*$-fragment.

Exercise 9.5. [**Finite Model Property**]
A set of formulas $\mathcal{F}$ is said to have the *finite model property* if for all $F \in \mathcal{F}$, the following two statements are equivalent:

1. F is satisfiable.
2. F has a finite model.

Give a decision procedure for satisfiability of any such set of formulas.

Homework 9.1. [Reduction] (8 points)
Consider the fragment of (closed) formulas of the form $\forall x_1 \ldots \forall x_n\ F$ where F involves no predicates besides equality, but arbitrary function symbols. We want to study a reduction which yields a decision procedure for this class of formulas.

For instance, let $F = (x_1 = x_2 \to f(f(x_1)) = f(g(x_2)))$. We index the occurrences of each function symbol from the inside out:

$$x_1 = x_2 \to \overbrace{f(\underbrace{f(x_1)}_{f_1})}^{f_2} = \overbrace{f(\underbrace{g(x_2)}_{g_1})}^{f_3}$$

and introduce a fresh variable for each instance. We add constraints which capture the congruence properties for all function symbols involved, and replace terms in the original formula by variables. This yields:

$$\begin{aligned}&(x_1 = x_{f_1} \to x_{f_1} = x_{f_2} \land \\ &x_{f_1} = x_{g_1} \to x_{f_2} = x_{f_3} \land \\ &x_1 = x_{g_1} \to x_{f_1} = x_{f_3}) \ \to \\ &(x_1 = x_2 \to x_{f_2} = x_{f_3})\end{aligned}$$

1. Explain how this construction can be used to obtain a procedure for deciding *validity* of formulas from the given fragment.
2. Give a formal description of the reduction.
3. Prove correctness of the reduction step in your decision procedure.

Homework 9.2. [FOL without Function Symbols] (6 points)
Describe an algorithm that transforms any formula (in FOL with equality) into an equisatisfiable formula (in FOL with equality) that does not use function symbols.

Hints: Functions can be modelled as relations satisfying some additional properties. Don't forget to deal with constants, i.e., functions with arity 0. A similar transformation as in the previous exercise might be helpful.

Homework 9.3. [Football] (6 points)
The 2018 football world cup is approaching. Germany's coach is explaining the tactics and the sentiments in the team:

- Every forward player will be in the starting lineup.
- All players in the starting lineup have nothing against each other.
- Every player has something against some other player.

Formalize the above facts as a formula in first-order logic.

1. Is the formula satisfiable? Give a model or a resolution proof.
2. A journalist inferred that every forward player has something against some non-forward player. Is this inference correct? Proof or counterexample!

Hint: The set of players in the starting lineup is a subset of the set of players.