Logics Exercise		
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SS 2017	Exercise Sheet 10	04.07.2017

Submission of homework: Before tutorial on 11.07.2017. You have to do the homework yourself; no teamwork allowed.

Exercise 10.1. [Sequent Calculus]
Prove the following formulas in sequent calculus, or give a countermodel that falsifies the formula.

1. $\neg \exists x P(x) \to \forall x \neg P(x)$
2. $(\forall x (P \lor Q(x))) \to (P \lor \forall x Q(x))$
3. $\forall x \exists y P(x, y) \to \exists y \forall x P(x, y)$

Exercise 10.2. [Counterexamples from Sequent Calculus]
Consider the following invalid statement: $\exists x P(x) \to \forall x P(x)$. Try to prove this statement in sequent calculus and derive a countermodel from the (incomplete) proof tree.

Exercise 10.3. [Substitution in Sequent Calculus]
Prove that $\vdash_G \Gamma \Rightarrow \Delta$ implies $\vdash_G \Gamma[t/x] \Rightarrow \Delta[t/x]$, where, for a set of formulas Γ, we define $\Gamma[t/x]$ to be $\{F[t/x] \mid F \in \Gamma\}$, i.e. free occurrences of x are replaced by t. Give two different proofs:

1. A syntactic proof, transforming the proof tree of $\vdash_G \Gamma \Rightarrow \Delta$.
2. A semantic proof, using correctness and completeness of $\vdash_G$.

Exercise 10.4. [Natural Deduction]
Prove the following formula using natural deduction.

$$\neg(\forall x(\exists y(\neg P(x) \land P(y))))$$

Homework 10.1. **[Counterexamples from Sequent Calculus]** (4 points)
Recall Exercise 10.2. We derived a countermodel from an incomplete proof tree. Now consider the statement $\forall x P(x) \rightarrow \neg P(x)$.

1. What happens when trying to prove the validity of this formula in sequent calculus?
2. How can we derive a countermodel from the proof tree?
3. Is there a smaller countermodel?

Homework 10.2. **[Proofs]** (16 points)
Prove the following statements using both natural deduction and sequent calculus if they are valid, or give a countermodel otherwise.

1. $\neg\forall x \exists y \forall z (\neg P(x, z) \wedge P(z, y))$
2. $\forall x \forall y \forall z (P(x, x) \wedge (P(x, y) \wedge P(y, z) \rightarrow P(x, z)))$
3. $\exists x (P(x) \rightarrow \forall x P(x))$

Caution: While you are free to carry out the sequent calculus proofs in Logitext, note that application of $\forall L$ and $\exists R$ delete the principal formula. You have to select "Contract" first before instantiating the principal formula.