Logics Exercise

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SS 2017 | Exercise Sheet 9 | 27.06.2017

Submission of homework: Before tutorial on 04.07.2017. You have to do the homework yourself; no teamwork allowed.

Exercise 9.1. **[Models of the $\exists^*\forall^*$ Class]**
Consider the $\exists^*\forall^*$ class, i.e. formulas of the form

$$\exists x_1 \dots \exists x_n \; \forall y_1 \dots \forall y_m \; F$$

where F is quantifier-free and contains no function symbols. Show that such a formula has a model iff it has a model of size n (assuming $n \geq 1$). What happens if we allow equality in F?

Solution:
If the formula has a model of size n, it obviously has a model. It remains to show the opposite direction. Any model has to assign values from the universe to the existentially quantified x_i. Construct a new model that only contains those x_i. This is still a model, because F contains no more existential quantifiers.

Adding (the encoding of) equality does not change the shape of the formula. But computing the quotient structure might reduce the size of the model, hence the statement has to modified to read "a model of size at most n".

Exercise 9.2. **[Miniscoping]**
In the lecture, we proved that deciding unsatisfiability of monadic FOL formulas can be reduced to deciding unsatisfiability of formulas from the $\exists^*\forall^*$ fragment by using miniscoping.

Prove the lemma that after miniscoping, no nested quantifiers remain.

Solution:
We prove by induction on the structure of the formula that after miniscoping, for each subformula of the form $\forall x F$ resp. $\exists x F$, F is a disjunction resp. conjunction of literals, each literal containing x free.

The only interesting cases are the quantifier cases. Assume we have a formula of the form $\exists x F$, such that no miniscoping rules are applicable, and by induction hypothesis, below quantifiers in F there are only disjunctions/conjunctions of literals containing the bound variable.

As no miniscoping rules are applicable, F must be a conjunction of literals and quantified formulas, such that each conjunct contains x free. So assume F contains a quantified formula, i.e., $F = \ldots \wedge Qy.F' \wedge \ldots$. By induction hypothesis, F' is a disjunction/conjunction of literals, each literal containing y free. However, as we are in the monadic fragment, a literal can contain at most one free variable. Thus, F' cannot contain x free, which is a contradiction to F containing quantifiers. Thus, F only contains literals, and thus has the desired shape.

The case for $\forall x F$ is similar.

Exercise 9.3. [$\exists^*\forall^*$ **with Equality**]
Show that unsatisfiability of formulas from the $\exists^*\forall^*$ fragment with equality is decidable.

Hint: Reduce it to the $\exists^*\forall^*$-fragment without equality.

Solution:
Applying the reduction of equality to non-equality from the lecture only inserts some (isolated) $\forall$-quantifiers, thus preserving the $\exists^*\forall^*$-fragment.

Exercise 9.4. [$\exists^*\forall^2\exists^*$]
Show how to reduce deciding unsatisfiability of formulas from the $\exists^*\forall^2\exists^*$-fragment to deciding unsatisfiability of formulas from the $\forall^2\exists^*$-fragment.

Solution:
Using skolemization for the outer existential quantifiers preserves satisfiability, and replaces variables by skolem constants, i.e., introduces no function symbols of arity > 0. The resulting formula is obviously in the $\forall^2\exists^*$-fragment.

Exercise 9.5. [**Finite Model Property**]
A set of formulas $\mathcal{F}$ is said to have the *finite model property* if for all $F \in \mathcal{F}$, the following two statements are equivalent:

1. F is satisfiable.
2. F has a finite model.

Give a decision procedure for satisfiability of any such set of formulas.

Solution:
Run the following routines in parallel:

1. Resolution (to check unsatisfiability).
2. Enumerate all finite models (to check satisfiability).

If the formula F is unsatisfiable, resolution will terminate. The result of the decision procedure is "unsatisfiable". If it is satisfiable, resolution might not terminate, but because of the finite model property, F will have a finite model that will be enumerated eventually. The result is "satisfiable".

Homework 9.1. [Reduction] (8 points)
Consider the fragment of (closed) formulas of the form $\forall x_1 \ldots \forall x_n\ F$ where F involves no predicates besides equality, but arbitrary function symbols. We want to study a reduction which yields a decision procedure for this class of formulas.

For instance, let $F = (x_1 = x_2 \to f(f(x_1)) = f(g(x_2)))$. We index the occurrences of each function symbol from the inside out:

$$x_1 = x_2 \to \overbrace{f(\underbrace{f(x_1)}_{f_1})}^{f_2} = \overbrace{f(\underbrace{g(x_2)}_{g_1})}^{f_3}$$

and introduce a fresh variable for each instance. We add constraints which capture the congruence properties for all function symbols involved, and replace terms in the original formula by variables. This yields:

$$\begin{aligned}
&(x_1 = x_{f_1} \to x_{f_1} = x_{f_2} \land \\
&x_{f_1} = x_{g_1} \to x_{f_2} = x_{f_3} \land \\
&x_1 = x_{g_1} \to x_{f_1} = x_{f_3}) \ \to \\
&(x_1 = x_2 \to x_{f_2} = x_{f_3})
\end{aligned}$$

1. Explain how this construction can be used to obtain a procedure for deciding *validity* of formulas from the given fragment.
2. Give a formal description of the reduction.
3. Prove correctness of the reduction step in your decision procedure.

Homework 9.2. [FOL without Function Symbols] (6 points)
Describe an algorithm that transforms any formula (in FOL with equality) into an equisatisfiable formula (in FOL with equality) that does not use function symbols.

Hints: Functions can be modelled as relations satisfying some additional properties. Don't forget to deal with constants, i.e., functions with arity 0. A similar transformation as in the previous exercise might be helpful.

Homework 9.3. [Football] (6 points)
The 2018 football world cup is approaching. Germany's coach is explaining the tactics and the sentiments in the team:

- Every forward player will be in the starting lineup.
- All players in the starting lineup have nothing against each other.
- Every player has something against some other player.

Formalize the above facts as a formula in first-order logic.

1. Is the formula satisfiable? Give a model or a resolution proof.
2. A journalist inferred that every forward player has something against some non-forward player. Is this inference correct? Proof or counterexample!

Hint: The set of players in the starting lineup is a subset of the set of players.