Propositional Logic
Compactness

Compactness Theorem

Theorem

A set S of formulas is satisfiable
iff every finite subset of S is satisfiable.

Equivalent formulation:

A set S of formulas is unsatisfiable
iff some finite subset of S is unsatisfiable.

Proof

$\Rightarrow$: If S is satisfiable then every finite subset of S is satisfiable.

Trivial.

$\Leftarrow$: If every finite subset of S is satisfiable then S is satisfiable.

We prove that S has a model.

Proof

Terminology: $\mathcal{A}$ is a $b_1, \dots, b_n$ model of T
(where $b_1, \dots, b_n \in \{0,1\}^*$ and T is a set of formulas)
if $\mathcal{A}(A_i) = b_i$ (for $i = 1, \dots, n$) and $\mathcal{A} \models T$.

Define an infinite sequence $b_1, b_2, \dots$ recursively as follows:

$$b_{n+1} \quad = \quad \text{some } b \in \{0,1\} \text{ s.t. all finite } T \subseteq S \text{ have a } b_1, \dots, b_n, b \text{ model.}$$

Claim 1: For all n, all finite $T \subseteq S$ have a $b_1, \dots, b_n$ model.
Proof by induction on n.

Case $n = 0$: because all finite $T \subseteq S$ are satisfiable.

Case $n + 1$: We need to show that a suitable b exists.
Proof by contradiction. Assume there is no suitable b.
Then there is a finite $T_0 \subseteq S$ that has no $b_1, \dots, b_n, 0$ model (0)
and there is a finite $T_1 \subseteq S$ that has no $b_1, \dots, b_n, 1$ model (1).
Therefore $T_0 \cup T_1$ has no $b_1, \dots, b_n$ model $\mathcal{A}$:
$\mathcal{A}(A_{n+1}) = 0$ contradicts (0), $\mathcal{A}(A_{n+1}) = 1$ contradicts (1).
But by IH: $T_0 \cup T_1$ has a $b_1, \dots, b_n$ model — Contradiction!

Proof

Define $\mathcal{B}(A_i) = b_i$ for all i.

Claim 2: $\mathcal{B} \models S$
We show $\mathcal{B} \models F$ for all $F \in S$.
Let m be the maximal index of all atoms in F.
By Claim 1, $\{F\}$ has a $b_1, \ldots, b_m$ model $\mathcal{A}$.
Hence $\mathcal{B} \models F$ because $\mathcal{A}$ and $\mathcal{B}$ agree on all atoms in F.

Corollary

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If $S \models F$ then there is a finite subset $M \subseteq S$ such that $M \models F$.