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SS 2018 — Exercise Sheet 4 — 30.04.2018

Submission of homework: Before tutorial on 08.05.2018. Until further notice, homework has to be submitted in groups of two students.

Exercise 4.1. [Atomic Cut]
Let A be an atomic formula. Prove that if $\vdash_G \Gamma \Rightarrow A, \Delta$ and $\vdash_G A, \Gamma \Rightarrow \Delta$, then $\vdash_G \Gamma \Rightarrow \Delta$.

Solution:
Let D_1 and D_2 be the derivations of the assumptions. Proof by induction on the depth of D_1. Case analysis: What is the last proof rule in D_1?

Ax There are two subcases:

1. A is the principal formula: $D_1 = \dfrac{}{\underbrace{A, \Gamma'}_{\Gamma} \Rightarrow A, \Delta}$

   Contract D_2: $A, \Gamma' \Rightarrow \Delta$

2. A is not principal: $D_1 = \dfrac{}{\underbrace{B, \Gamma'}_{\Gamma} \Rightarrow A, \underbrace{B, \Delta'}_{\Delta}}$

   $\vdash_G \Gamma \Rightarrow \Delta$ by Ax

∧L $D_1 = \dfrac{F, G, \Gamma' \overset{D_1'}{\Rightarrow} A, \Delta}{\underbrace{F \wedge G, \Gamma'}_{\Gamma} \Rightarrow A, \Delta}$

$$\dfrac{\dfrac{D_1' \qquad \dfrac{D_2}{A, F, G, \Gamma' \Rightarrow \Delta}\,\wedge\mathrm{L}^{-1}}{F, G, \Gamma' \Rightarrow \Delta}\,\mathrm{IH}}{\Gamma \Rightarrow \Delta}\,\wedge\mathrm{L}$$

Other cases are similar.

Exercise 4.2. [Natural Deduction]
Prove the following formulas by natural deduction:

1. $(F \wedge G) \wedge H \rightarrow F \wedge (G \wedge H)$
2. $(F \vee G) \vee H \rightarrow F \vee (G \vee H)$
3. $\neg(F \wedge G) \rightarrow (\neg F \vee \neg G)$

Solution:

Exercise 4.2

1)

$$
\begin{prooftree}
\AxiomC{$[(F\wedge G)\wedge H]^1$}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{$(F\wedge G)\wedge H$}
\LeftLabel{$\wedge E$}
\UnaryInfC{$F\wedge G$}
\LeftLabel{$\wedge E$}
\UnaryInfC{F}
\AxiomC{$[(F\wedge G)\wedge H]^1$}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{$(F\wedge G)\wedge H$}
\LeftLabel{$\wedge E$}
\UnaryInfC{$F\wedge G$}
\LeftLabel{$\wedge E$}
\UnaryInfC{G}
\AxiomC{$[(F\wedge G)\wedge H]^1$}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{$(F\wedge G)\wedge H$}
\RightLabel{$\wedge E$}
\UnaryInfC{H}
\RightLabel{$\wedge I$}
\BinaryInfC{$G\wedge H$}
\RightLabel{$\wedge I$}
\BinaryInfC{$F\wedge(G\wedge H)$}
\RightLabel{$\rightarrow I\ (1)$}
\UnaryInfC{$(F\wedge G)\wedge H \rightarrow F\wedge(G\wedge H)$}
\end{prooftree}
$$

2)

$$
\begin{prooftree}
\AxiomC{$[(F\vee G)\vee H]^1$}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{$(F\vee G)\vee H$}
\AxiomC{$[F\vee G]^2$}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{$F\vee G$}
\AxiomC{$[F]^4$}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{F}
\RightLabel{$\vee I$}
\UnaryInfC{$F\vee(G\vee H)^{(4)}$}
\AxiomC{$[G]^5$}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{G}
\RightLabel{$\vee I$}
\UnaryInfC{$G\vee H$}
\RightLabel{$\vee I$}
\UnaryInfC{$^{(5)}F\vee(G\vee H)$}
\LeftLabel{$\vee E$}
\TrinaryInfC{$F\vee(G\vee H)\ (2)$}
\AxiomC{$[H]^3$}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{H}
\RightLabel{$\vee I$}
\UnaryInfC{$G\vee H$}
\RightLabel{$\vee I$}
\UnaryInfC{$F\vee(G\vee H)\ (3)$}
\RightLabel{$\vee E$}
\TrinaryInfC{$F\vee(G\vee H)$}
\RightLabel{$\rightarrow I\ (1)$}
\UnaryInfC{$(F\vee G)\vee H \rightarrow F\vee(G\vee H)$}
\end{prooftree}
$$

3)

$$
\begin{prooftree}
\AxiomC{$[\neg(F\wedge G)]^1$}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{$\neg(F\wedge G)$}
\AxiomC{(similar)}
\noLine
\UnaryInfC{F}
\AxiomC{$[\neg(\neg F\vee\neg G)]^2$}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{$\neg(\neg F\vee\neg G)$}
\AxiomC{$[\neg G]^3$}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{$\neg G$}
\RightLabel{$\vee I$}
\UnaryInfC{$\neg F\vee\neg G$}
\RightLabel{$\neg E$}
\BinaryInfC{$\bot$}
\RightLabel{$\bot\ (3)$}
\UnaryInfC{G}
\RightLabel{$\wedge I$}
\BinaryInfC{$F\wedge G$}
\RightLabel{$\neg E$}
\BinaryInfC{$\bot$}
\RightLabel{$\bot\ (2)$}
\UnaryInfC{$\neg F\vee\neg G$}
\RightLabel{$\rightarrow I\ (1)$}
\UnaryInfC{$\neg(F\wedge G) \rightarrow (\neg F\vee\neg G)$}
\end{prooftree}
$$

Exercise 4.3. **[Classical Reasoning]**
We replace rule $\bot$ of the calculus of natural deduction by either one of the following rules:

- $\dfrac{}{F \lor \neg F}$ (law of excluded middle)
- $\dfrac{\neg\neg F}{F}$ (double negation elimination)

Additionally, we add the rule $\dfrac{\bot}{F}$ $(\bot E)$. Show that the calculus of natural deduction remains complete in both cases.

Solution:
We want to show that the $\bot$ rule can be derived from either of the two other alternatives. Thus we assume that there is a proof of the form

$$\begin{array}{c}\neg F\\ \vdots\\ \bot\end{array}$$

and we need to show that we then can also prove F.

- $$\dfrac{\dfrac{}{F \lor \neg F}\,\text{(law of excluded middle)} \qquad [F]^1 \qquad \dfrac{\begin{array}{c}[\neg F]^1\\ \vdots\\ \bot\end{array}}{F}\,\bot E}{F}\,\lor E\ (1)$$

- $$\dfrac{\dfrac{\begin{array}{c}[\neg F]^1\\ \vdots\\ \bot\end{array}}{\neg\neg F}\,\neg I\ (1)}{F}\,\text{(double negation elimination)}$$

Homework 4.1. **[Natural Deduction]** (10 points)
Prove the following formulas by natural deduction (as specified in the lecture):

1. $((A \to B) \to A) \to A$
2. $(\neg G \to F) \to (\neg F \to G)$
3. $\neg\neg\neg F \to \neg F$ (without using the $\perp$ rule, but the $\perp E$ rule from Exercise 4.3 is allowed)

Homework 4.2. **[Substitution]** (10 points)
Assume that there are proofs for $\vdash_N G \to G'$ and $\vdash_N G' \to G$. Construct the proof for $\vdash_N F[G/A] \to F[G'/A]$.