Semantics of Programming Languages

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1 Introduction

1.1 Aims of the lecture series

Descriptive: how to describe semantics
Analytic: how to use semantics to analyze programming language
how to analyze tools: is a compiler correct?
how to analyze programs

$$\begin{array}{rcl} c & ::= & \mathbf{skip} \\ & & \mid x := a \\ & & \mid c;\ c \\ & & \mid \mathbf{if}\ b\ \mathbf{then}\ c\ \mathbf{else}\ c \\ & & \mid \mathbf{while}\ b\ \mathbf{do}\ c \end{array}$$

Overview

1. **Operational Semantics**
   - (command, start state) $\rightarrow$ final state
   - inference rules $\dfrac{B(b)\sigma = true \quad (c,\sigma) \rightarrow \sigma' \quad (\mathbf{while}\ b\ \mathbf{do}\ c, \sigma') \rightarrow \sigma''}{(\mathbf{while}\ b\ \mathbf{do}\ c, \sigma) \rightarrow \sigma''}$

     B evaluation function
   - Type System, Compiler } correctness
2. **Denotational Semantics**
   - S: Syntax $\rightarrow$ Semantics
   - command $\rightarrow$(state $\nrightarrow$ state)

     $\nrightarrow$ partial function
   - $S(\mathbf{while}\ b\ \mathbf{do}\ c)\sigma = \begin{cases} \sigma \text{ if } B(b)\sigma = false \\ S(\mathbf{while}\ b\ \mathbf{do}\ c)(S(c)\sigma) \text{ otherwise} \end{cases}$
3. **Axiomatic Semantics (Hoare logic)**
   - {precondition} command {postcondition}
   - Example: $\{x := 0\} x := x+1 \{x = 2\}$ not correct!
   - correctness and completeness
   - application: runtime analysis of programs

1. $function\ f() :\ integer\ \{\ldots\};$
   $e : \mathbf{if}\ f() = f()\ \mathbf{then}\ 0\ \mathbf{else}\ 1$

   Q: how can e evaluate to 1?

   $$\{x := x+1;\ return\ x\}$$

2. $function\ f() :\ integer\ \{return f()\}$

   Q: Find e so that $e + f()$ and $f() + e$ differ.
   A: $e := \frac{1}{0}$

3.

$$\begin{array}{rl} x := 0 & \{x = 0\} \\ a[1] := 0 & \{a[1] = 0\} \\ a[a[1]] := 0 & \{a[a[1]] = 0\} \\ \{a[1] = 1 \wedge a[0] = 2\}\ a[a[1]] := 0 & \{a[a[1]] = 0\} \end{array}$$

$$\left.\begin{array}{l} addr(x) \\ addr(a[1]) \end{array}\right\} \quad \text{constant}$$

$$addr(a[a[1]]) \quad \text{can change!}$$

but works with $\{a[1] \neq 1\}$ as precondition

Q: Find context that distinguishes

$$x := 0;\ y := 1$$
and
$$y := 1;\ x := 0$$

var $z : integer$
proc $p($**var** $x, y : integer)\ \{\ldots\}$
$p(z, z)$
proc $p\ \{\ldots\}$
proc q $\{$**var** $x : int;\ x := 0;\ p\}$

Q: Can p and q be distinguished?
A: Possible Out-of-Memory-exception when calling q. p could access the stack.

proc $p($**proc** $r)$ $\{\ldots\}$ no side effects
proc q $\{$ **var** $x : int;$
proc $r\ \{x := x + 2\};$
$x := 0;\ p(r)$
if $even(x)$ **then** $\langle s\rangle\}$

Q: Is q equivalent to $\langle s\rangle$?
A: Yes, if p terminates or if $\langle s\rangle$ does not terminate.

Find context that distinguishes $1+2$ and $2+1$:

$$1+2*3 \neq 2+1*3$$
$$"1+2" \neq "2+1"$$

$A \Rightarrow B \equiv \neg A \vee B$
$A \Rightarrow (B \Rightarrow C) \equiv B \Rightarrow (A \Rightarrow C) \neq (A \Rightarrow B) \Rightarrow C$

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1.2 The WHILE language

Set	Meta-Var
$\mathbb{Z}$	i, j, m, n (constants)
var	x, y, z
$T = \{\text{tt}, \text{ff}\}$	

Syntax

Arithmetic expressions (*Aexp*)

$$a ::= n \mid x \mid a_1 \; aop \; a_2$$

where $aop = \{+, -, *\}$

Boolean expressions (*Bexp*)

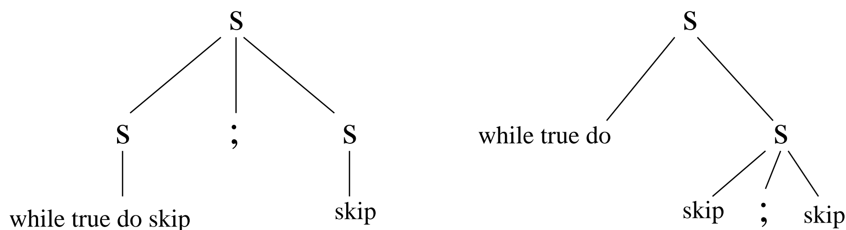
$$b ::= \textbf{true} \mid \textbf{false} \mid a \; rop \; a \mid b \; bop \; b \mid \neg b$$

where $rop = \{=, \leq\}$ and $bop = \{\wedge, \vee, \Rightarrow, \ldots\}$

Statements (*Stm*)[1]

$$s ::= \textbf{skip} \mid x := a \mid s; \; s \mid \textbf{if } b \textbf{ then } s \textbf{ else } s \mid \textbf{while } b \textbf{ do } s$$

Concrete syntax: grammars for strings, should be *unambiguous*
Stm is ambiguous: **while true do skip** ; **skip**



Abstract syntax: grammar for trees
Use (,) do disambiguate:

(**while true do skip**); **skip**

while true do (**skip** ; **skip**)

Compare tree level and string level of "2+1*3" example

The State

$$\begin{aligned} \text{State} &= \text{mapping from } \textbf{var} \text{ to } \mathbb{Z} \\ \Sigma &= \textbf{var} \rightarrow \mathbb{Z} \text{ (Meta-var: } \sigma) \end{aligned}$$

Updating the state

$$\begin{aligned} \sigma[x \mapsto n] &= \text{"}\sigma \text{ with } x \text{ bound to } n\text{"}^{2} \\ (\sigma[x \mapsto n])(y) &= \begin{cases} n & \text{if } x = y \\ \sigma(x) & \text{if } x \neq y \end{cases} \end{aligned}$$

[1] [WINSKEL93]: "command"

Concrete states: $\{x \mapsto 5, y \mapsto 7\}$

Semantics of expressions

$\mathcal{A} : Aexp \to (\Sigma \to \mathbb{Z}) \quad (Aexp \times \Sigma \to \mathbb{Z})$
$\mathcal{B} : Bexp \to \Sigma \to T$

$$\begin{array}{rcl} \mathcal{A}[n]\sigma & = & n \text{ (because } n \text{ is constant!)} \\ \mathcal{A}[x]\sigma & = & \sigma(x) \\ \mathcal{A}[a_1\ op\ a_2]\sigma & = & [aop](\mathcal{A}[a_1]\sigma, \mathcal{A}[a_2]\sigma) \end{array}$$

For each other operator $\otimes \in aop$, $[\otimes]$ is of type $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$

Convention
[]: Syntax →Semantics

$$\begin{array}{rcl} \mathcal{B}[\mathbf{true}\ ]\sigma & = & \text{tt} \\ \mathcal{B}[a_1\ rop\ a_2]\sigma & = & [rop](\mathcal{A}[a_1]\sigma, \mathcal{A}[a_2]\sigma) \\ & & \text{where } [rop] : \mathbb{Z} \times \mathbb{Z} \to T \\ \mathcal{B}[b_1\ bop\ b_2]\sigma & = & [bop](\mathcal{B}[b_1]\sigma, \mathcal{B}[b_2]\sigma) \\ & & \text{where } [bop] : T \times T \to T \\ & & \text{and } [\wedge] = \begin{array}{c|c|c} & \text{tt} & \text{ff} \\ \hline \text{tt} & \dots & \dots \\ \text{ff} & \dots & \dots \end{array} \\ \mathcal{B}[\neg b]\sigma & = & \begin{cases} \text{tt} & \text{if } \mathcal{B}[b]\sigma = \text{ff} \\ \text{ff} & \text{if } \mathcal{B}[b]\sigma = \text{tt} \end{cases} \end{array}$$

$$\begin{array}{rcl} \mathcal{A}[x+5]\{x \overset{\sigma}{\mapsto} z\} & = & [+](\mathcal{A}[x]\sigma, \mathcal{A}[5]\sigma) \\ & = & [+](7,5) = 12 \end{array}$$

Fact: $\mathcal{A}$ and $\mathcal{B}$ are unique total functions because

1. each syntactic case is covered
2. the arguments of $\mathcal{A}$ / $\mathcal{B}$ on the right hand side are strictly smaller than on the left hand side (i.e. computation terminates)
3. for each expression, only one clause for $\mathcal{A}$ / $\mathcal{B}$ applies

Definition of $\mathcal{A}$ / $\mathcal{B}$ is by *primitive recursion*.
Corresponding proof principle: *structural induction*.

[2] [WINSKEL93]: $\sigma[n/x]$

2 Operational Semantics

Aim: abstract mathematical model of computation steps
Method: inference rules (derivation rules)

$$\frac{\bullet \quad \bullet}{\bullet} \rightsquigarrow \text{Prolog}$$

2.1 Big-step semantics

(Natural semantics, Evaluation semantics)

Idea: relation (transition relation)
$\langle statement, state\rangle \to state$
$\langle s, \sigma\rangle \to \sigma'$

Formally: $\to \ \subseteq Stm \times \Sigma \times \Sigma$
Notation: $(s, \sigma, \sigma') \in \to$ is an abbreviation to $\langle s, \sigma\rangle \to \sigma'$

Rules for $\to$

$$\left.\begin{array}{c} \overline{\langle \mathbf{skip}, \sigma\rangle \to \sigma} \\ \\ \langle x := a, \sigma\rangle \to \sigma[x \mapsto \mathcal{A}[a]\sigma] \end{array}\right\} \text{axioms}$$

$$\frac{\langle s_1, \sigma_1\rangle \to \sigma_2 \qquad \langle s_2, \sigma_2\rangle \to \sigma_3}{\langle s_1; s_2, \sigma\rangle \to \sigma_3}$$

$$\frac{\mathcal{B}[b]\sigma = \mathrm{tt} \qquad \langle s_1, \sigma\rangle \to \sigma'}{\langle \mathbf{if}\ b\ \mathbf{then}\ s_1\ \mathbf{else}\ s_2, \sigma\rangle \to \sigma'} \text{ analogously for ff}, s_2$$

w = **while** b **do** s

$$\frac{\mathcal{B}[b]\sigma = \mathrm{ff}}{\langle w, \sigma\rangle \to \sigma} \qquad \frac{\mathcal{B}[b]\sigma = \mathrm{tt} \qquad \langle s, \sigma\rangle \to \sigma' \qquad \langle w, \sigma'\rangle \to \sigma''}{\langle w, \sigma\rangle \to \sigma''}$$

Remarks:

1. General format of rules

$$\frac{premise_1 \quad \dots \quad premise_n}{conclusion} \text{ (name)}$$

2. Axiom = rule without premises
3. Rules can be composed to form derivation trees.
4. Directly executable in Prolog.

$$\frac{\overline{\langle x := 1, \sigma_0\rangle \to \sigma_1} \qquad \overline{\langle y := x, \sigma_1\rangle \to \sigma_2}}{\langle x := 1;\ y := x, \sigma_0\rangle \to \sigma_2}$$

$$\begin{array}{lcl} \sigma_0 & = & \{x \mapsto 0, y \mapsto 0\} \\ \sigma_1 & = & \sigma_0[x \mapsto 1] = \{x \mapsto 1, y \mapsto 0\} \\ \sigma_2 & = & \sigma_1[y \mapsto x] = \{x \mapsto 1, y \mapsto 1\} \end{array}$$

$w = \textbf{while } x \neq 1 \textbf{ do } x := x+1$
$\sigma_n = \{x \mapsto n\}$

$$\dfrac{\mathcal{B}[x\neq 1]\sigma_0 = \text{tt} \qquad \langle x := x+1, \sigma_0\rangle \to \sigma_1 \qquad \dfrac{\mathcal{B}[x\neq 1]\sigma_1 = \text{ff}}{\langle w,\sigma_1\rangle \to \sigma_1}}{\langle w,\sigma_0\rangle \to \sigma_1}$$

$\langle w,\sigma_2\rangle \to$ has no final state
$\nexists \sigma'.\langle w,\sigma_2\rangle \to \sigma'$

Fact: In a big-step semantics, nontermination of s starting in σ is expressed by the non-existence of some σ' such that $\langle s,\sigma\rangle \to \sigma'$

Program equivalence with respect to $\to$

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$$s_1 \sim s_2 \iff \forall \sigma,\sigma'.\langle s_1,\sigma\rangle \to \sigma' \Leftrightarrow \langle s_2,\sigma\rangle \to \sigma'$$

Lemma:

$$w \sim \textbf{if } b \textbf{ then } s;\ w \textbf{ else skip } =: w'$$

Proof: Show $\langle w,\sigma\rangle \to \sigma' \Leftrightarrow \langle w',\sigma\rangle \to \sigma'$ by case distinction on $\mathcal{B}[b]\sigma$ and analysis of the last inference rule used.

1. $\mathcal{B}[b]\sigma = \text{ff}$

$$\dfrac{}{\langle w,\sigma\rangle \to \sigma} \Leftrightarrow \dfrac{\langle \textbf{skip},\sigma\rangle \to \sigma}{\langle w',\sigma\rangle \to \sigma}$$

Motto: Transformation of derivations

2. $\mathcal{B}[b]\sigma = \text{tt}$

$$\dfrac{\dfrac{D_1}{\langle s,\sigma\rangle \to \sigma''} \qquad \dfrac{D_2}{\langle w,\sigma''\rangle \to \sigma'}}{\langle w,\sigma\rangle \to \sigma'} \Leftrightarrow \dfrac{\dfrac{\dfrac{D_1}{\langle s,\sigma\rangle \to \sigma''} \qquad \dfrac{D_2}{\langle w,\sigma''\rangle \to \sigma'}}{\langle s;\ w,\sigma\rangle \to \sigma'}}{\langle w',\sigma\rangle \to \sigma'}$$

Theorem: WHILE is deterministic

$$\langle s,\sigma\rangle \to \sigma_1 \wedge \langle s,\sigma\rangle \to \sigma_2 \Rightarrow \sigma_1 = \sigma_2$$

Proof: by induction on s
1 case: $s = \textbf{while } b \textbf{ do } s_0$ and $\mathcal{B}[b]\sigma = \text{tt}$

$$\dfrac{\langle s_0,\sigma\rangle \to \sigma_1' \qquad \langle s,\sigma_1'\rangle \to \sigma_1}{\langle s,\sigma\rangle \to \sigma_1} \wedge \dfrac{\langle s_0,\sigma\rangle \to \sigma_2' \qquad \langle s,\sigma_2'\rangle \to \sigma_2}{\langle s,\sigma\rangle \to \sigma_2} \overset{I.H.}{\Rightarrow} \sigma_1' = \sigma_2' \overset{I.H.}{\Rightarrow} \sigma_1 = \sigma_2 \ \lightning$$

(no structural induction, as s is not smaller than s); but: rule induction over number of derivation steps.

Problem: while - rule is not compositional with respect to the syntax; **while** uses **while** .
$\left[\text{Compositional: } \mathcal{A}[a_1+a_2]\sigma = [''+''](\mathcal{A}[a_1]\sigma, \mathcal{A}[a_2]\sigma)\right]$

2.2 Rule induction - an introduction

A set of rules $\dfrac{a_1 \in X \quad \ldots \quad a_n \in X}{a \in X}$ defines X *inductively* as the least (smallest) set that satisfies all rules. (least = only those elements that are provably in the set = no junk)

Example: $0 \in N \quad \dfrac{n \in N}{n+1 \in N}$ defines $\mathbb{N}$

but $\mathbb{R}$ also satisfies these rules! $\mathbb{N}$ is the least set. Associated proof principle: rule induction.

Idea: $P(x)$ holds[3] for all $x \in N$ if P is *"preserved by all rules"*: $P(0)$ and $P(n) \Rightarrow P(n+1)$

In general: $P(a_1) \wedge \ldots \wedge P(a_n) \Rightarrow P(a)$

Proof principle:
$[x \in X \Rightarrow P(x)]$ if P is preserved by all rules $(\forall x \in X . P(x))$
Motivation: induction on the size of the derivation.

Some easy examples

$$0 \in N \qquad (R0)$$

$$\frac{n \in N}{n+1 \in N} \qquad (R1)$$

Lemma: $n \in N \Rightarrow \underbrace{n+n \in N}_{P(n)}$

Proof: P is preserved by $R0$ and $R1$:

$P(0) = (0+0 \in N)\checkmark$ (by R0)

$$\begin{array}{ccc} P(n) & \overset{?}{\Rightarrow} & P(n+1) \\ \shortparallel & & \shortparallel \\ n+n \in N & & (n+1)+(n+1) \in N \\ \searrow & & \nearrow \\ & (n+n)+1 \in N & \end{array}$$

Lemma: $m \in N \wedge n \in N \Rightarrow m+n \in N$

$$m \in N \Rightarrow (\forall n. \underbrace{n \in N \Rightarrow m+n \in N}_{P(m)})$$

Proof: by rule induction

$$\begin{array}{lll} R0 & : & P(0) = (n \in N \Rightarrow 0+n \in N)\checkmark \\ R1 & : & P(m) \overset{?}{=} P(m+1) \\ & & \text{Given: } n \in N \Rightarrow m+n \in N \quad \text{(I.H.)} \rightarrow P(m) \\ & & \text{Show: } n \in N \Rightarrow (m+1)+n \in N \end{array}$$

[3] "is provable"

Transitive reflexive closure

Given a relation $\to$, define $\xrightarrow{*}$:

$$\frac{}{x \xrightarrow{*} x} \quad R0$$

$$\frac{x \xrightarrow{*} y \qquad \overbrace{y \to z}^{\text{side condition}}}{x \xrightarrow{*} z} \quad R1$$

Rule induction for $\xrightarrow{*}$

$x \xrightarrow{*} y \Rightarrow P(x,y)$ if

1. $(\forall x)P(x,x)$
2. $P(x,y) \wedge y \to z \Rightarrow P(x,z)$

Lemma: $x \to y \Rightarrow x \xrightarrow{*} y$ (derived rule)[4]

Proof: $$\text{R1}\ \frac{\text{R0}\ \frac{}{x \xrightarrow{*} x} \qquad x \to y}{x \xrightarrow{*} y}$$

Theorem: $x \to y \wedge y \xrightarrow{*} z \Rightarrow x \xrightarrow{*} z \equiv y \xrightarrow{*} z \Rightarrow \underbrace{(x \to y \Rightarrow x \xrightarrow{*} z)}_{P(y,z)}$

Proof: by rule induction

Rename $b \xrightarrow{*} c \Rightarrow \underbrace{(a \to b \Rightarrow a \xrightarrow{*} c)}_{P(b,c)}$

$R0$: $P(x,x) = (a \to x \Rightarrow a \xrightarrow{*} x)$ Lemma ✓

$R1$: $P(x,y) \wedge y \to z \overset{?}{\Rightarrow} P(x,z)$

Given: $a \to x \Rightarrow a \xrightarrow{*} y$
Given: $y \to z$
$\Big\} \overset{R1}{\Rightarrow} a \xrightarrow{*} z$

Show: $a \to x \Rightarrow a \xrightarrow{*} z$

$$\text{R}\ \frac{a_1 \in X \quad \ldots \quad a_n \in X \quad \overbrace{C_1 \ldots C_m}^{\text{side conditions; dont't mention } X}}{a \in X}$$

P is preserved by R if

$$P(a_1) \wedge \ldots \wedge P(a_n) \wedge C_1 \wedge \ldots \wedge C_m \Rightarrow P(a)$$

$\langle _,_ \rangle \to _ \subseteq Stm \times \Sigma \times \Sigma$
$\langle s,\sigma \rangle \to \sigma' = (s,\sigma,\sigma') \in \langle _,_ \rangle \to _$

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[4] no induction, only composition of rules

To show: $\forall s, \sigma, \sigma' : \langle s, \sigma \rangle \to \sigma' \Rightarrow P(s, \sigma, \sigma')$

We have to prove

- $\forall \sigma . P(\textbf{skip}\ , \sigma, \sigma)$
- $\forall x, a, \sigma . P\left(x := a, \sigma, \sigma\left[x \mapsto [a]_\sigma\right]\right)$
- $\forall s_1, s_2, \sigma, \sigma', \sigma'' . P(s_1, \sigma, \sigma'') \wedge P(s_2, \sigma'', \sigma') \Rightarrow P(s_1;\ s_2, \sigma, \sigma')$
- $\forall b, s_1, s_2, \sigma, \sigma' . P(s_1, \sigma, \sigma') \wedge [b]_\sigma = \text{tt} \Rightarrow P(\textbf{if}\ b\ \textbf{then}\ s_1 \textbf{else}\ s_2, \sigma, \sigma')$
  $\forall b, s_1, s_2, \sigma, \sigma' . P(s_2, \sigma, \sigma') \wedge [b]_\sigma = \text{ff} \Rightarrow P(\textbf{if}\ b\ \textbf{then}\ s_1 \textbf{else}\ s_2, \sigma, \sigma')$
- $\forall b, s, \sigma . [b]_\sigma = \text{ff} \Rightarrow P(\textbf{while}\ b\ \textbf{do}\ s, \sigma, \sigma)$
  $\forall b, s, \sigma, \sigma', \sigma'' . [b]_\sigma = \text{tt} \wedge P(w, \sigma'', \sigma') \wedge P(s, \sigma, \sigma'') \Rightarrow P(w, \sigma, \sigma')$

WHILE is deterministic:

$$\forall s, \sigma, \sigma', \sigma'' . \langle s, \sigma \rangle \to \sigma' \wedge \langle s, \sigma \rangle \to \sigma'' \Rightarrow \sigma' = \sigma'' \tag{1}$$

Proof: by rule induction on $\to$

- transforming (1):

$$\forall s, \sigma, \sigma' . \langle s, \sigma \rangle \to \sigma' \Rightarrow \underbrace{(\forall \sigma'' . \langle s, \sigma \rangle \to \sigma'' \Rightarrow \sigma' = \sigma'')}_{P(s,\sigma,\sigma')}$$

- show $P(\textbf{skip}\ , \sigma, \sigma) = \forall \sigma'' . \langle \textbf{skip}\ , \sigma \rangle \to \sigma'' \Rightarrow \sigma = \sigma''$ (by **skip** - rule)
- show ...
- show $[b]_\sigma = \text{tt} \wedge P(s, \sigma, \sigma_0) \wedge P(w, \sigma_0, \sigma') \Rightarrow P(w, \sigma, \sigma')$

  assume $\begin{array}{l} \forall \sigma'' . \langle s, \sigma \rangle \to \sigma'' \Rightarrow \sigma_0 = \sigma'' \quad (IH_s) \\ \forall \sigma'' . \langle w, \sigma_0 \rangle \to \sigma'' \Rightarrow \sigma' = \sigma'' \quad (IH_w) \end{array}$

  show $\forall \sigma'' . \langle w, \sigma \rangle \to \sigma'' \Rightarrow \sigma' = \sigma''$ (σ'' fixed, but arbitrary.)
  assume $\langle w, \sigma \rangle \to \sigma''$
  show $\sigma' = \sigma''$

  Derivation of $\langle w, \sigma \rangle \to \sigma''$

$$\frac{[b]_\sigma = \text{tt} \qquad \overbrace{\langle s, \sigma \rangle \to \sigma_1}^{\mathcal{S}} \qquad \langle w, \sigma_1 \rangle \to \sigma''}{\langle w, \sigma \rangle \to \sigma''}$$

  from $\mathcal{S}$ and IH_s we have $\sigma_0 = \sigma_1 \Rightarrow \langle s, \sigma \rangle \to \sigma_0$ and $\langle w, \sigma_0 \rangle \to \sigma''$
  with IH_w we have $\sigma' = \sigma''$. ✓

2.3 Structural operational semantics

(transition semantics, small-step semantics)

Idea: not one big step, but many small steps.

Format: $\langle s,\sigma\rangle \rightarrow_1 \gamma$ where

- either γ is of the form $\langle s',\sigma'\rangle$;
  then the execution has not terminated and s' is the "remaining" statement to execute
- or γ is of the form σ';
  then the execution has terminated in state σ'

Definition: $\langle s,\sigma\rangle$ is *stuck* if $\nexists\gamma.\langle s,\sigma\rangle \rightarrow_1 \gamma$

Rules for $\rightarrow_1$:

- $\langle \textbf{skip}\ ,\sigma\rangle \rightarrow_1 \sigma$
- $\langle x := a,\sigma\rangle \rightarrow_1 \sigma[x \mapsto \mathcal{A}[a]\sigma]$
- $\dfrac{\langle s_1,\sigma\rangle \rightarrow_1 \langle s_1',\sigma'\rangle}{\langle s_1;s_2,\sigma\rangle \rightarrow_1 \langle s_1';s_2,\sigma'\rangle}$
- $\dfrac{\langle s_1,\sigma\rangle \rightarrow_1 \sigma'}{\langle s_1;s_2,\sigma\rangle \rightarrow_1 \langle s_2,\sigma'\rangle}$
- $\dfrac{\mathcal{A}[b]\sigma = \text{tt} \ /\ \text{ff}}{\langle \textbf{if}\ b\ \textbf{then}\ s_1\ \textbf{else}\ s_2,\sigma\rangle \rightarrow_1 \langle s_1,\sigma\rangle \ /\ \langle s_2,\sigma\rangle}$
- $\dfrac{}{\langle w,\sigma\rangle \rightarrow_1 \langle \textbf{if}\ b\ \textbf{then}\ s;w\ \textbf{else}\ \textbf{skip}\ ,\sigma\rangle}$

Definition:
A *derivation sequence* for $\langle s,\sigma\rangle = \gamma_0$ is either a finite sequence $\gamma_0 \rightarrow_1 \gamma_1 \rightarrow_1 \ldots \rightarrow_1 \gamma_k$ such that $\gamma_k \in \Sigma$ or γ_k is stuck (*termination*) or an infinite sequence $\gamma_0 \rightarrow_1 \gamma_1 \rightarrow_1 \ldots$ (*nontermination*).
Each step $\gamma_i \rightarrow_1 \gamma_{i+1}$ must be justified by a derivation tree. Usual notation $\xrightarrow{n}_1$ / $\xrightarrow{*}_1$ means n / finitely many steps of execution.

Example: $\sigma = \{x \mapsto 1, y \mapsto 2\}$

$$\begin{aligned}\langle (z := x; x := y); y := z,\sigma\rangle &\rightarrow_1 \langle x := y; y := z, \sigma[z \mapsto 1]\rangle \\ &\rightarrow_1 \langle y := z, \sigma[z \mapsto 1, x \mapsto 2]\rangle \\ &\rightarrow_1 \sigma[z \mapsto 1, x \mapsto 2, y \mapsto 1]\end{aligned} \tag{2}$$

Justify each step with derivation tree:

$$(2): \dfrac{\dfrac{\vdots \qquad \vdots}{\langle z := x; x := y,\sigma\rangle \rightarrow_1 \langle x := y,\sigma_z\rangle}}{\langle (z := x; x := y); y := z,\sigma\rangle \rightarrow_1 \langle x := y; y := z,\sigma_0\rangle}$$

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Example: $w = \textbf{while}\ x \neq 1\ \textbf{do}\ x := x+1 \quad \sigma_n = \{x \mapsto n\}$

$$\langle w,\sigma_0\rangle \quad \rightarrow_1 \quad \langle \underbrace{\textbf{if}\ x \neq 1\ \textbf{then}\ x := x+1; w\ \textbf{else}\ \textbf{skip}}_{\textbf{if}}\ ,\sigma_0\rangle$$

$$\begin{array}{rcl} \langle \textbf{if}, \sigma_0 \rangle & \to_1 & \langle x := x+1; w, \sigma_0 \rangle \\ \langle x := x+1; w, \sigma_0 \rangle & \to_1 & \langle w, \sigma_1 \rangle \\ \langle w, \sigma_1 \rangle & \to_1 & \langle \textbf{if}, \sigma_1 \rangle \\ \langle \textbf{if}, \sigma_1 \rangle & \to_1 & \langle \textbf{skip}, \sigma_1 \rangle \\ \langle \textbf{skip}, \sigma_1 \rangle & \to_1 & \sigma_1 \end{array}$$

$$\begin{array}{l} \langle w, \sigma_2 \rangle \to_1 \langle \textbf{if}, \sigma_2 \rangle \to_1 \langle x := x+1; w, \sigma_2 \rangle \to_1 \langle w, \sigma_3 \rangle \\ \langle w, \sigma_3 \rangle \to_1 \dots \end{array}$$

Lemma: There are no stuck $\langle s, \sigma \rangle$.

Proof: by induction on s (structural induction)

- $s = \textbf{skip}$ $\to$ not stuck
- $s = s_1; s_2$ IH: $\exists \gamma. \langle s_1, \sigma \rangle \to_1 \gamma$
  Case distinction on γ
  - $\gamma = \sigma' \Rightarrow \langle s_1; s_2, \sigma \rangle \to_1 \langle s_2, \sigma' \rangle$ ✓
  - $\gamma = \langle s_1', \sigma' \rangle \Rightarrow \langle s_1; s_2, \sigma \rangle \to_1 \langle s_1'; s_2, \sigma \rangle$ ✓

Predominant proof principle for $\xrightarrow{*}_1$: *induction* on the length of the derivation sequence.

Lemma: If $\langle s_1; s_2, \sigma \rangle \xrightarrow{n}_1 \sigma''$ then there exist σ', i, j such that

$$\langle s_1, \sigma \rangle \xrightarrow{i}_1 \sigma' \text{ and } \langle s_2, \sigma' \rangle \xrightarrow{j}_1 \sigma'' \text{ and } n = i + j$$

Proof: by induction on n
$n = 0:$ ✓
$n \to n+1$: assume the lemma holds for n, then prove for $n+1$:

$$\langle s_1; s_2, \sigma \rangle \to_1 \gamma \xrightarrow{n}_1 \sigma''$$

$\to$ Case distinction on first step

1. $$\frac{\langle s_1, \sigma \rangle \to_1 \sigma_1}{\langle s_1; s_2, \sigma \rangle \to_1 \underbrace{s_2, \sigma_1}_{\gamma}}$$
   Then the lemma holds for $\sigma' = \sigma_1, i = 1, j = n$ ✓
2. $$\frac{\langle s_1, \sigma \rangle \to_1 \langle s_1', \sigma_1 \rangle}{\langle s_1; s_2, \sigma \rangle \to_1 \langle \underbrace{s_1', s_2}_{\gamma}, \sigma_1 \rangle}$$
   By IH:
   $\langle s_1', \sigma_1 \rangle \xrightarrow{i_x}_1 \sigma_x$

$\langle s_2, \sigma_x \rangle \overset{j_x}{\rightarrow}_1 \sigma'' \quad i_x + j_x = n$

Then the lemma holds for $\sigma' = \sigma_x, i = i_x + 1, j = j_x$
$\langle s_1, \sigma \rangle \overset{i}{\rightarrow}_1 \sigma'$ because $\langle s_1, \sigma \rangle \rightarrow_1 \langle s_1', \sigma_1 \rangle \overset{i_x}{\rightarrow}_1 \sigma'$ ✓

Theorem: (Agreement of big-step and small-step semantics)

$$\langle s, \sigma \rangle \rightarrow \sigma' \Leftrightarrow \langle s, \sigma \rangle \overset{*}{\rightarrow}_1 \sigma'$$

This implies

- Termination in one semantics iff. termination in the other semantics, and with same result
- Looping in one semantics iff. looping in the other semantics.

Proof: "⇒": exercise (Übungsblatt 3)
"⇐": $\langle s, \sigma \rangle \overset{n}{\rightarrow}_1 \sigma' \Rightarrow \langle s, \sigma \rangle \rightarrow \sigma'$

Proof by induction on n:
IH: Assume that it works for all $m < n$, then show it holds for n

$n > 0$, because $\langle s, \sigma \rangle \overset{0}{\rightarrow}_1 \sigma'$ is not possible. Case distinction on s:

- $s = \textbf{skip}$: $\sigma = \sigma', \langle \textbf{skip}, \sigma \rangle \rightarrow \sigma$ ✓
- $s = (x := a) : \sigma' = \sigma[x \mapsto \mathcal{A}[a]\sigma], \langle x := a, \sigma \rangle \rightarrow \sigma[x \mapsto \mathcal{A}[a]\sigma]$
- $s = s_1; s_2$: show $\langle s_1; s_2, \sigma \rangle \rightarrow \sigma'$

$$\begin{array}{lll} \text{lemma} & \Rightarrow & \exists \sigma'', i, j. \langle s_1, \sigma \rangle \overset{i}{\rightarrow}_1 \sigma'' \wedge \langle s_2, \sigma'' \rangle \overset{j}{\rightarrow}_1 \sigma' \wedge n = i + j \\ & \overset{IH}{\Rightarrow} & \langle s_1, \sigma \rangle \rightarrow \sigma'' \wedge \langle s_2, \sigma'' \rangle \rightarrow \sigma' \\ & \overset{;-rule}{\Rightarrow} & \langle s_1; s_2, \sigma \rangle \rightarrow \sigma' \quad ✓ \end{array}$$

- $s = \textbf{if}$ ✓ (similar as ;)
- $w = \textbf{while}\ b\ \textbf{do}\ s_0 = s$

$$\langle w, \sigma \rangle \rightarrow_1 \langle \underbrace{\textbf{if}\ b\ \textbf{then}\ s_0; w\ \textbf{else skip}}_{w'}, \sigma \rangle \overset{n-1}{\rightarrow}_1 \sigma'$$

$$\begin{array}{ll} \overset{IH}{\Rightarrow} & \langle w', \sigma \rangle \rightarrow \sigma' \\ \Rightarrow & \langle s, \sigma \rangle \rightarrow \sigma' \text{ because } w \sim w' \end{array}$$

2.4 Extensions of the WHILE language

- Nondeterminism
- Parallelism
- Blocks
- Procedures

} comparison of big/small step semantics

2.4.1 Nondeterminism

New statement: s_1 **or** s_2

Example:

$$\underbrace{\langle \overbrace{x := 1}^{s_1} \textbf{ or } \overbrace{(x := 2; x := x+4)}^{s_{24}}}_{s_{124}}, \sigma\rangle \rightarrow \begin{cases} x = 1 \\ x = 6 \end{cases}$$

Big-step rules

$$\frac{\langle s_1, \sigma\rangle \rightarrow \sigma'}{\langle s_1 \textbf{ or } s_2, \sigma\rangle \rightarrow \sigma'} \qquad \frac{\langle s_2, \sigma\rangle \rightarrow \sigma'}{\langle s_1 \textbf{ or } s_2, \sigma\rangle \rightarrow \sigma'}$$

Example:

$$\frac{\overline{\langle s_1, \sigma\rangle \rightarrow \sigma[x \mapsto 1]}}{\langle s_{124}, \sigma\rangle \rightarrow \sigma[x \mapsto 1]} \qquad \frac{\dfrac{\overline{\langle s_2, \sigma\rangle \rightarrow \sigma[x \mapsto 2]} \quad \overline{\langle s_4, \sigma[x \mapsto 2]\rangle \rightarrow \sigma[x \mapsto 6]}}{\langle s_{24}, \sigma\rangle \rightarrow \sigma[x \mapsto 6]}}{\langle s_{124}, \sigma\rangle \rightarrow \sigma[x \mapsto 6]}$$

$$\langle x := 1 \textbf{ or } (\underbrace{\textbf{while true do skip}}_{\Omega}), \sigma\rangle \rightarrow \ ? \ \ \sigma[x \mapsto 1]$$

Small-step rules:

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$$\begin{array}{lcl} \langle s_1 \textbf{ or } s_2, \sigma\rangle & \rightarrow_1 & \langle s_1, \sigma\rangle \\ \langle s_1 \textbf{ or } s_2, \sigma\rangle & \rightarrow_1 & \langle s_2, \sigma\rangle \end{array}$$

Example:

$$\begin{array}{lcl} \langle s_{124}, \sigma\rangle & \rightarrow_1 & \langle s_{24}, \sigma\rangle \rightarrow_1 \langle s_4, \sigma[x \mapsto 2]\rangle \rightarrow_1 \sigma[x \mapsto 6] \\ \langle s_{124}, \sigma\rangle & \rightarrow_1 & \langle s_1, \sigma\rangle \rightarrow_1 \sigma[x \mapsto 1] \end{array}$$

$$\begin{array}{lcl} \langle x := 1 \textbf{ or } \Omega, \sigma\rangle & \rightarrow_1 & \langle x := 1, \sigma\rangle \rightarrow_1 \sigma[x \mapsto 1] \\ \langle x := 1 \textbf{ or } \Omega, \sigma\rangle & \rightarrow_1 & \langle \Omega, \sigma\rangle \rightarrow_1 \langle \textbf{if true then skip }; \Omega \textbf{ else skip }, \sigma\rangle \\ & \rightarrow_1 & \langle \textbf{skip }; \Omega, \sigma\rangle \rightarrow_1 \langle \Omega, \sigma\rangle \rightarrow_1 \ldots \end{array}$$

Comparison:

- big-step semantics hides nontermination:

$$x := 1 \textbf{ or } \Omega \quad \sim \quad x := 1$$

- small-step semantics does not hide termination:

$$\begin{array}{rl} x := 1 & \text{has no infinite derivation sequence} \\ x := 1 \textbf{ or } \Omega & \text{has an infinite derivation sequence} \end{array}$$

2.4.2 Parallelism

New statement $s_1 \parallel s_2$
Semantics: interleaving execution (s_1 and s_2 "alternate")

Small-step rules

$$\frac{\langle s_1, \sigma\rangle \rightarrow_1 \langle s_1', \sigma'\rangle}{\langle s_1 \parallel s_2, \sigma\rangle \rightarrow_1 \langle s_1' \parallel s_2, \sigma'\rangle} \qquad \frac{\langle s_1, \sigma\rangle \rightarrow_1 \sigma'}{\langle s_1 \parallel s_2, \sigma\rangle \rightarrow_1 \langle s_2, \sigma'\rangle}$$

$$\frac{\langle s_2, \sigma\rangle \rightarrow_1 \langle s_2', \sigma'\rangle}{\langle s_1 \parallel s_2, \sigma\rangle \rightarrow_1 \langle s_1 \parallel s_2', \sigma'\rangle} \qquad \frac{\langle s_2, \sigma\rangle \rightarrow_1 \sigma'}{\langle s_1 \parallel s_2, \sigma\rangle \rightarrow_1 \langle s_1, \sigma'\rangle}$$

Example: $s_{124} := x := 1 \parallel (x := 2; x := x + 4) \qquad \Rightarrow x = 1, 5, 6$

$$\langle s_{124}, \sigma\rangle \quad \rightarrow_1 \quad \langle s_{24}, \sigma[x \mapsto 1]\rangle \rightarrow_1 \ldots \rightarrow_1 \sigma[x \mapsto 6]$$

$$\begin{array}{rcl} \langle s_{124}, \sigma\rangle & \rightarrow_1 & \langle x := 1 \parallel x := x + 4, \sigma[x \mapsto 2]\rangle \\ & \rightarrow_1 & \langle x := 1, \sigma[x \mapsto 6]\rangle \rightarrow_1 \sigma[x \mapsto 1] \end{array}$$

$$\begin{array}{rcl} \langle s_{124}, \sigma\rangle & \rightarrow_1 & \langle x := 1 \parallel x := x + 4, \sigma[x \mapsto 2]\rangle \\ & \rightarrow_1 & \langle x := x + 4, \sigma[x \mapsto 1]\rangle \rightarrow_1 \sigma[x \mapsto 5] \end{array}$$

Big-step rules

$$\frac{?}{\langle s_1 \parallel s_2, \sigma\rangle \rightarrow \sigma'}$$

Comparison:

- Big-steps hide intermediate states $\Rightarrow$ parallelism impossible
- Small-steps show intermediate states $\Rightarrow$ parallelism possible

A language extension is called *modular* if

- new rules are added, but
- old rules stay unchanged

or is modular: $\rightarrow$ and $\rightarrow_1$
$\parallel$ is modular: $\rightarrow_1$

Not all language extensions are modular!

2.4.3 Blocks (Local variables)

New statement: $\{\mathbf{var}\ x := a; s\}$ **Idea:** x is local variable in s.

Example:

$$\langle x := 1;\ \{\mathbf{var}\ x := 5; \mathbf{skip}\ \}, \sigma\rangle \rightsquigarrow \sigma[x \mapsto 1]$$

$$\langle y := 1;\ \{\mathbf{var}\ x := 5; \mathbf{skip}\ \}, \sigma\rangle \rightsquigarrow \sigma[y \mapsto 1]$$

Big-step rules

$$\frac{\langle x := a; s, \sigma\rangle \to \sigma'}{\langle\{\mathbf{var}\ x := a; s\}, \sigma\rangle \to \sigma'[x \mapsto \sigma(x)]}$$

Small-step rules

$$\langle\{\mathbf{var}\ x := a; s\}, \sigma\rangle \to_1 \langle x := a; s; x := \underbrace{\sigma(x)}_{\in\mathbb{Z}}, \sigma\rangle$$

2.4.4 Procedures

New statements: $\{\mathbf{var}\ x := a; s\}$
$\{\mathbf{proc}\ (p = s_1); s_2\}$
$\mathbf{call}\ p$
Parameterless procedures!

Example:

```
{ var x:=0;
  { proc p = x:=2*x;
    { proc q = call p;
      { var x:=5;
        { proc p = x:=x+1;
          call q; y:=x
}}}}}
```

Final value of y? $\to 5$!

Static scope: Name x/p refers to the closest declaration of x/p in the *text*.
Dynamic scope: Name x/p refers to the most recent definition of x/p during *execution*.

- static scope for **var** and for **proc** : $y = 5$
- dynamic scope for **var** and static scope for **proc** :

$$x := 0; x := 5; x := 2 * x; y := x$$

$\to y = 10, x = 0$ after 4th "}"; at the end of "**call** $q; y := x$", x is not defined.

- dynamic scope for **var** and **proc** :

$$x : 0; x := 5; \overset{x:=x+1}{(x := 6)}; y := x \to y = 6$$

Dynamic scope for proc and var

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Procedure environment: $Penv = \underbrace{Pname}_{name} \to \underbrace{Stm}_{body}$

$pe \in Penv$
Format of transitions: $pe \vdash \langle s, \sigma \rangle \rightarrow \sigma'$
"In the context of pe, execution of s starting in σ leads to σ'."

Rules for **while** -language: add $pe \vdash$ everywhere.

$$\frac{pe \vdash \langle s_1, \sigma_1 \rangle \rightarrow \sigma_2 \qquad pe \vdash \langle s_2, \sigma_2 \rangle \rightarrow \sigma_3}{pe \vdash \langle s_1; s_2, \sigma_1 \rangle \rightarrow \sigma_3}$$

$$\frac{pe \vdash \langle s, \sigma[x \mapsto \mathcal{A}[a]\sigma] \rangle \rightarrow \sigma'}{pe \vdash \langle \{\mathbf{var}\ x := a;\ s\}, \sigma \rangle \rightarrow \sigma'[x \mapsto \sigma(x)]}$$

$$\frac{pe[p \mapsto s_1] \vdash \langle s_2, \sigma \rangle \rightarrow \sigma'}{pe \vdash \langle \{\mathbf{proc}\ p = s_1; s_2\}, \sigma \rangle \rightarrow \sigma'}$$

$$\frac{pe \vdash \langle pe(p), \sigma \rangle \rightarrow \sigma'}{pe \vdash \langle \mathbf{call}\ p, \sigma \rangle \rightarrow \sigma'}$$

```
s0  { var x:=0;
s1    { proc p = x:=2*x;
s2      { proc q = call p;
s3        { var x:=5;
s4          { proc p = x:=x+1;
s5              call q; y:=x;
    }}}}}
```

Example: $\{\} \vdash \langle s_0, \sigma \rangle \rightarrow ? \qquad \sigma_i = \sigma[x \mapsto i]$

$$\begin{array}{c}
\{\} \vdash \langle s_0, \sigma \rangle \rightarrow \sigma_0[y \mapsto 6] \\ \hline
\{\} \vdash \langle s_1, \sigma_0 \rangle \rightarrow \sigma_0[y \mapsto 6] \\ \hline
\{p \mapsto x := 2 * x\} \vdash \langle s_2, \sigma_0 \rangle \rightarrow \sigma_0[y \mapsto 6] \\ \hline
\{p \mapsto x := 2 * x, q \mapsto \mathbf{call}\ p\} \vdash \langle s_3, \sigma_0 \rangle \rightarrow \sigma_0[y \mapsto 6] \\ \hline
\{p \mapsto x := 2 * x, q \mapsto \mathbf{call}\ p\} \vdash \langle s_4, \sigma_5 \rangle \rightarrow \sigma_6[y \mapsto 6] \\ \hline
\{p \mapsto x := x + 1, q \mapsto \mathbf{call}\ p\} \vdash \langle s_5, \sigma_5 \rangle \rightarrow \sigma_6[y \mapsto 6] \\ \hline
\{p \mapsto x := x + 1, q \mapsto \mathbf{call}\ p\} \vdash \langle \mathbf{call}\ q, \sigma_5 \rangle \rightarrow \sigma_6 \qquad \{\%\} \vdash \langle y := x, \sigma_6 \rangle \rightarrow \sigma_6[x \mapsto 6] \\ \hline
\{p \mapsto x := x + 1, q \mapsto \mathbf{call}\ p\} \vdash \langle \mathbf{call}\ p, \sigma_5 \rangle \rightarrow \sigma_6 \\ \hline
\{p \mapsto x := x + 1, q \mapsto \mathbf{call}\ p\} \vdash \langle x := x + 1, \sigma_5 \rangle \rightarrow \sigma_6
\end{array}$$

Static scope for proc, dynamic scope for var

Idea: remember the static environment (i.e. the program text) for each procedure

$$Penv = Pname \rightarrow \underbrace{(Stm \times Penv)}_{\text{closure}}$$

use $\stackrel{fin}{\rightarrow}$ instead of $\nrightarrow$ to make this work

(s, pe) is closed: procedure names in s are defined in pe ($\leftrightarrow$ open formula: free variables)

The rules: **WHILE** -language and **var** unchanged

$$\frac{pe[p \mapsto (s_1, pe)] \vdash \langle s_2, \sigma \rangle \rightarrow \sigma'}{pe \vdash \langle \{\mathbf{proc}\ p = s_1; s_2\}, \sigma \rangle \rightarrow \sigma'}$$

$$\frac{pe' \vdash \langle s, \sigma \rangle \rightarrow \sigma'}{pe \vdash \langle \mathbf{call}\ p, \sigma \rangle \rightarrow \sigma'} \qquad \text{if } pe(p) = (s, pe')$$

Example: $s = \{\textbf{proc } f = \ldots \textbf{call } f \ldots ; \textbf{call } f\}$

$$\cfrac{\cfrac{\{\} \vdash \langle s, \sigma\rangle \to}{\left\{f \mapsto (\ldots \textbf{call } f \ldots, \{\})\right\} \vdash \langle \textbf{call } f, \sigma\rangle \to}}{\{\} \vdash \langle \ldots \textbf{call } f \ldots\rangle \to \quad \lightning}$$

cannot call f in empty pe!

Corrected call rule

$$\frac{pe'[p \mapsto pe(p)] \vdash \langle s, \sigma\rangle \to \sigma'}{pe \vdash \langle \textbf{call } p, \sigma\rangle \to \sigma'} \text{ if } pe(p) = (s, pe')$$

Mini-example

```
s0  { proc p = x:=0;
s1    { proq q = call p;
s2      { proc p = x:=1;
s3        { call q }}}
```

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\{\} \vdash \langle s_0, \sigma\rangle \to \sigma[x \mapsto 0]}{\underbrace{\{p \mapsto (x := 0, \{\})\}}_{pe_1} \vdash \langle s_1, \sigma\rangle \to \sigma[x \mapsto 0]}}{\underbrace{\{q \mapsto (\textbf{call } p, pe_1), p \to \ldots\}}_{pe_2} \vdash \langle s_2, \sigma\rangle \to \sigma[x \mapsto 0]}}{\{q \mapsto (\textbf{call } p, pe_1), p \mapsto (x := 1, pe_2)\} \vdash \langle \textbf{call } q, \sigma\rangle \to \sigma[x \mapsto 0]}}{pe_1[q \mapsto (\textbf{call } p, pe_1)] \vdash \langle \textbf{call } p, \sigma\rangle \to \sigma[x \mapsto 0]}}{\vdash \langle x := 0, \sigma\rangle \to \sigma[x \mapsto 0] \ \Uparrow}$$

An inductive definition of *Penv*:

- $\{\} \in Penv$
- $\dfrac{pe_1, pe_2 \in Penv \qquad s \in Stm \qquad p \in Pname}{pe_1[p \mapsto (s, pe_2)]}$

An alternative: recursive environments!
Example:

$$p \mapsto (s, \{\ldots, p \mapsto (s, \{\ldots, p \mapsto \ldots\ldots$$

$$\underbrace{pe = \{p \mapsto (s, pe)\}}_{pointer!}$$

$$\frac{pe' \vdash \langle s_2, \sigma\rangle \to \sigma'}{pe \vdash \langle\{\textbf{proc } p = s_1; s_2\}, \sigma\rangle \to \sigma'} \qquad \text{where } pe' = pe[p \mapsto (s_1, pe')] \text{ (recursive definition)}$$

$\Rightarrow$ simpler call rule

$$\frac{pe' \vdash \langle s, \sigma\rangle \to \sigma'}{pe \vdash \langle \textbf{call } p, \sigma\rangle \to \sigma'} \qquad \text{if } pe(p) = (s, pe')$$

Static scope for proc and var

Idea: separate variable names (*Var*) from storage locations (*Loc*)

- *Loc*: some infinite set
- Variable environment: $Venv = Var \stackrel{fin}{\longrightarrow} Loc \quad \ni ve$ [5]
- $Store = Loc \stackrel{fin}{\longrightarrow} \mathbb{Z} \quad \ni \sigma$
- $\sigma \circ ve \in Var \to \mathbb{Z}$
- $Penv = Pname \stackrel{fin}{\longrightarrow} (Stm \times Venv \times Penv)$
- (s, ve, pe): variable x in s refers to location $ve(x)$
- procedure p in s refers to the body $pe(p)$

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Format of transitions: $ve, pe \vdash \langle s, \sigma \rangle \to \sigma'$
Assignment: $ve, pe \vdash \langle x := a, \sigma \rangle \to \sigma[\underbrace{ve(x)}_{\in Loc} \mapsto \mathcal{A}[a](\underbrace{\sigma \circ ve}_{\in \Sigma})]$
Remaining while-rules: add *ve* everywhere

Convention: execution starts with *ve* and σ such that

$$\begin{aligned} range(ve) &= \{l \mid \exists x. ve(x) = l\} \subseteq dom(\sigma) \\ &= \{l \mid \sigma(l) \text{ is defined}\} \end{aligned} \tag{3}$$

Rules will guarantee that (3) is invariant under each transition.
Final value of global variable x: $\sigma'(ve(x)) \quad (x \in dom(ve))$

Rules for **var** :

$$\frac{ve[x \mapsto l], pe \vdash \langle s, \sigma[l \mapsto \mathcal{A}[a](\sigma \circ ve)] \rangle \to \sigma'}{ve, pe \vdash \langle \{\textbf{var } x := a;\ s\}, \sigma \rangle \to \sigma'}$$

where l is some "new" location $\notin dom(\sigma)$
why $\to \sigma'$ and not $\sigma'[l \mapsto undefined]$?
We allow garbage in σ': l is "unreachable".

$$\frac{ve, pe[p \mapsto (s_1, ve, pe)] \vdash \langle s_2, \sigma \rangle \to \sigma'}{ve, pe \vdash \langle \{\textbf{proc } p = s_1;\ s_2\}, \sigma \rangle \to \sigma'}$$

$$\frac{ve', pe'[p \mapsto pe(p)] \vdash \langle s, \sigma \rangle \to \sigma'}{ve, pe \vdash \langle \textbf{call } p, \sigma \rangle \to \sigma'} \qquad \text{if } pe(p) = (s, ve', pe')$$

2.5 A compiler for WHILE

Source language: **WHILE**
Target language: Idealized assembly language (IAL)
Instructions:

- *ASSIGN Var Aexp*
- *JMP* $\mathbb{Z}$ (relative jump)
- *JMPF Bexp* $\mathbb{Z}$ (jump if *Bexp* is false)

Semantics of IAL: Relation on program counter × State

Notation: $\boxed{P \vdash \langle i, \sigma \rangle \to \langle j, \sigma' \rangle}$
where

[5] $(A \stackrel{fin}{\longrightarrow} B$ = all functions $A \rightharpoonup B$ that are defined for only finitely many arguments)

- P: list of instructions
- $i, j \in \mathbb{Z}, \sigma, \sigma' \in \Sigma$

"Execution of instruction i of P in state σ leads to the new program counter j and new state σ'."

Notation: $P_n = (n+1)^{th}$ instruction in P, i.e. P_0 is first instruction. $++$ = concatenation of lists.

Rules:

$$\frac{P_n = ASSIGN\ x\ a}{P \vdash \langle n, \sigma \rangle \to \langle n+1, \sigma[x \mapsto \mathcal{A}[a]\sigma] \rangle}$$

$$\frac{P_n = JMP\ d}{P \vdash \langle n, \sigma \rangle \to \langle n+d, \sigma \rangle}$$

$$\frac{P_n = JMPF\ b\ d \qquad \mathcal{B}[b]\sigma = \text{tt}\ /\ \text{ff}}{P \vdash \langle n, \sigma \rangle \to \langle n+1\ /\ n+d, \sigma \rangle}$$

Notation: $P \vdash c_0 \xrightarrow{*} c_n$ iff. there are $c_1, \ldots, c_{n-1}$ such that

- $P \vdash c_0 \to c_1$ and
- ... and
- $P \vdash c_{n-1} \to c_n$

Execution stops when pc negative or too large because then P_n is undefined.

$$\begin{array}{rcl}
comp : Stm & \to & \text{instruction list} \\
comp\ \textbf{skip} & = & [] \\
comp\ (x := a) & = & [ASSIGN\ x\ a] \\
comp\ (s_1; s_2) & = & comp(s_1) ++ comp(s_2) \\
comp\ (\textbf{if}\ b\ \textbf{then}\ s_1\ \textbf{else}\ s_2) & = & \textbf{let}\ \begin{array}{l} is_1 = comp\ s_1 \\ is_2 = comp\ s_2 \end{array} \\
 & & \textbf{in}\ [JMPF\ b\ (|is_1| + 2)] ++ is_1 ++ [JMP |is_2| + 1] ++ is_2 \\
comp\ (\textbf{while}\ b\ \textbf{do}\ s) & = & \textbf{let}\ is = comp\ s \\
 & & \textbf{in}\ [JMPF\ b\ |is| + 2] ++ is ++ [JMP - (|is| + 1)]
\end{array}$$

Theorem:

$$\langle s, \sigma \rangle \to \sigma' \Rightarrow comp\ s \vdash \langle 0, \sigma \rangle \xrightarrow{*} \langle |comp\ s|, \sigma' \rangle$$

Proof: by induction on $\langle s, \sigma \rangle \to \sigma'$ (rule induction)

case $s = s_1; s_2$: $\langle s_1, \sigma \rangle \to \overline{\sigma}, \langle s_2, \overline{\sigma} \rangle \to \sigma'$

$$\frac{\langle s_1, \sigma \rangle \to \overline{\sigma} \qquad \langle s_2, \overline{\sigma} \rangle \to \sigma' \qquad \leftarrow \text{assume this}}{\langle s, \sigma \rangle \to \sigma' \Rightarrow Q(s, \sigma, \sigma') \qquad \leftarrow \text{show this}}$$

Abbreviation: $P_i = comp\ s_i$

IH_1: $P_1 \vdash \langle 0, \sigma \rangle \xrightarrow{*} \langle |P_1|, \overline{\sigma} \rangle$

IH_2: $P_2 \vdash \langle 0, \overline{\sigma} \rangle \xrightarrow{*} \langle |P_2|, \sigma' \rangle$

To show: $P_1 ++ P_2 \vdash \langle 0,\sigma\rangle \xrightarrow{*} \langle |P_1|+|P_2|,\sigma'\rangle$

Lemma:
$P \vdash \langle i,\sigma\rangle \xrightarrow{n} \langle j,\sigma'\rangle \Rightarrow P ++ P' \vdash \langle i,\sigma\rangle \xrightarrow{n} \langle j,\sigma'\rangle$
$P \vdash \langle i,\sigma\rangle \xrightarrow{n} \langle j,\sigma'\rangle \Rightarrow P' ++ P \vdash \langle |P'|+i,\sigma\rangle \xrightarrow{n} \langle |P'|+j,\sigma\rangle$

Proof: by induction on n followed by a case distinction on the first rule used.

$$\left.\begin{array}{l} IH_1 + \text{Lemma: } P_1 ++ P_2 \vdash \langle 0,\sigma\rangle \xrightarrow{*} \langle |P_1|,\overline{\sigma}\rangle \\ IH_2 + \text{Lemma: } P_1 ++ P_2 \vdash \langle |P_1|,\overline{\sigma}\rangle \xrightarrow{*} \langle |P_1|+|P_2|,\sigma'\rangle \end{array}\right\} \Rightarrow P_1 ++ P_2 \vdash \langle 0,\sigma\rangle \xrightarrow{*} \langle |P_1|+|P_2|,\sigma'\rangle \text{ qed.}$$

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Theorem: $\langle s,\sigma\rangle \to \sigma' \Rightarrow comp\ s \vdash \langle 0,\sigma\rangle \xrightarrow{*} \langle |comp\ s|,\sigma'\rangle$

Theorem: Let $c = comp\ s$. $c \vdash \langle 0,\sigma\rangle \xrightarrow{*} \langle |c|,\sigma'\rangle \Rightarrow \langle s,\sigma\rangle \to \sigma'$

Proof: by induction on s (because $comp$ is defined recursively)
Case $s = \textbf{while}\ b\ \textbf{do}\ s_0 \Rightarrow$

$$comp\ s \quad = \quad * \ [JMPF\ b\ *] ++ c_0 ++ [JMP\ *]\ *$$

$$c_0 = comp\ s_0$$

We prove $c \vdash \langle 0,\sigma\rangle \xrightarrow{n} \langle |c|,\sigma'\rangle \Rightarrow \langle s,\sigma\rangle \to \sigma'$ by induction on n. We assume it holds for all $m < n$ and prove it for n. Case distinction:

1. $\mathcal{B}[b]\sigma = \text{ff} \Rightarrow \sigma' = \sigma \Rightarrow \langle s,\sigma\rangle \longrightarrow\!\!\!\!\longrightarrow \sigma'$

2. $\mathcal{B}[b]\sigma = \text{tt} \Rightarrow c \vdash \langle 0,\sigma\rangle \to \langle 1,\sigma\rangle \xrightarrow{n-1} \langle |c|,\sigma'\rangle$

   $\exists\overline{\sigma}, n_1, n_2 . c_0 \vdash \langle 0,\sigma\rangle \xrightarrow{n_1} \langle |c_0|,\overline{\sigma}\rangle \wedge$
   $c \vdash \langle |c_0|+1,\overline{\sigma}\rangle \xrightarrow{n_2} \langle |c|,\sigma'\rangle \wedge n_1 + n_2 = n-1$

   (n_1: number of executions of s_0; n_2: number of executions for next iterations[6])

Definition: A list of instructions is *closed* iff. all jumps in c have a "destination" $\in \{0,\ldots,|c|\}$. ("No jump leaves c")

Lemma: (1) Let c be closed.

$$\begin{array}{ll} & p ++ c ++ p' \vdash \langle |p|+i,\sigma\rangle \xrightarrow{n} \langle j,\sigma'\rangle \\ & \text{where } 0 \le i \le |c| \wedge \neg(|p| \le j \le |p|+|c|) \\ \Rightarrow & \exists\overline{\sigma}, n_1, n_2 . c \vdash \langle i,\sigma\rangle \xrightarrow{n_1} \langle |c|,\sigma'\rangle \wedge \\ & p ++ c ++ p' \vdash \langle |p|+|c|,\overline{\sigma}\rangle \xrightarrow{n_2} \langle j,\sigma'\rangle \wedge \\ & n_1 + n_2 = n \end{array}$$

[6] by Lemma(1) and Lemma(2)

Proof: by induction on n.

Lemma: (2) *comp s* is closed.

Proof: by induction on s.

$$
\begin{array}{lcl}
\text{Lemma(1) and Lemma(2)} & \overset{IH_s}{\Rightarrow} & \langle s_0, \sigma\rangle \rightarrow \overline{\sigma} \wedge c \vdash \langle 0, \overline{\sigma}\rangle \overset{n_2-1}{\rightarrow} \langle |c|, \sigma'\rangle \\
 & \overset{IH_1}{\Rightarrow} & \langle s, \overline{\sigma}\rangle \rightarrow \sigma' \\
 & \overset{\text{by while-rule}}{\Rightarrow} & \langle s, \sigma\rangle \rightarrow \sigma' \qquad qed.
\end{array}
$$

3 Types

11/19/2001

Classification

- dynamic type system (out)
  - runtime test if types are ok
  - no help for program development
  - example: LISP
- static type system (in!)
  - analysis of types at compile time
  - only limited precision (e.g. int/bool, but not $\{0,\ldots,42\}$)
  - extremely helpful for program development
  - special case of program analysis and verification

Aim: static, automatic, efficient, precise
guarantee absence of as many runtime errors as possible

3.1 Types for WHILE

Types: **bool** and **int**
$\rightarrow$ Now allowed: $B := I < 0$
$\rightarrow$ Not allowed: $I := J + 1; I := I > J$

Previously: *Aexp* and *Bexp*
Now: $Exp := Aexp \cup Bexp$; type system seperates.

Syntax of *Exp*:

$$e ::= n \mid \textbf{true} \mid \textbf{false} \mid X \mid e\ aop\ e \mid e\ bop\ e \mid e\ rop\ e \mid \neg e$$

"Raw expressions" (e.g. $10 \wedge \neg 5$)

Exp = set of all (raw/untyped) expressions

Modified stm syntax:

- $x := e$
- **if** e **then** ...
- **while** e **do** ...

$\Rightarrow$ New language $\textbf{WHILE}_\textbf{T}$ ("Typed **WHILE** ")

Assumption: "Every variable has a fixed type."

Possible interpretations:

1. Type is fixed for all programs
   (e.g. $I - N$: always integer)
2. Type must be declared in each program
   (e.g. **var** I : integer / **int** I / ...)
3. Type is inferred / computed by the "system"
   $\Rightarrow$ Type interference / Type reconstruction

1 und 2 are *type checking* methods.

3.2 A monomorphic type system

Types $\tau ::= int \mid bool$

Program syntax: $p ::= \underbrace{x_1 : \tau_1; \ldots ; x_n : \tau_n}_{variable\ declarations\ \Gamma} ; s$

Assumption: Γ never declares a variable twice.
*Mono*morphic: a variable has at most one type.

3.2.1 A typesystem for $\mathbf{WHILE_T}$

Types of expression

Want to say "In context Γ, expression e has type τ."

Notation: $\Gamma \vdash e : \tau$

Rules: $\Gamma \vdash n : int, \Gamma \vdash true\ /\ false :\ bool$

$$\frac{x : \tau \in \Gamma}{\Gamma \vdash x : \tau}$$

$$\frac{\Gamma \vdash e_1 : int \qquad \Gamma \vdash e_2 : int}{\Gamma \vdash e_1\ aop\ e2\ \ : \mathbf{int}} \qquad \text{analogously: bop}$$

$$\frac{\Gamma \vdash e_1 : int \qquad \Gamma \vdash e_2 : int}{\Gamma \vdash e_1\ rop\ e2\ \ : \mathbf{bool}} \qquad + \neg e$$

Example: $\Gamma = X : \mathbf{int}\ , Y : \mathbf{bool}$
$\Gamma \vdash (X = X) \wedge \neg Y : \mathbf{bool}$
not $\Gamma \vdash X = Y : \tau$ for any type!

Terminology: e is *type correct* (in context Γ) if e has a type, i.e. $\Gamma \vdash e : \tau$ for some τ.

Type system for statements

Notation: $\Gamma \vdash s$ ("Statement s is type correct in context Γ")

Rules:

$$\Gamma \vdash skip$$

$$\frac{x : \tau \in \Gamma \qquad \Gamma \vdash e : \tau}{\Gamma \vdash x := e}$$

$$\frac{\Gamma \vdash s_1 \qquad \Gamma \vdash s_2}{\Gamma \vdash s_1 ; s_2}$$

$$\frac{\Gamma \vdash b : \mathbf{bool} \qquad \Gamma \vdash s_1 \qquad \Gamma \vdash s_2}{\Gamma \vdash\ \mathbf{if}\ \mathrm{b}\ \mathbf{then}\ s_1\ \mathbf{else}\ s_2}$$

$$\frac{\Gamma \vdash b : \mathbf{bool} \qquad \Gamma \vdash s}{\Gamma \vdash \mathbf{while}\ b\ \mathbf{do}\ s}$$

Program $p = \Gamma; s$ is type correct iff. $\Gamma \vdash s$

Note: rules for $\Gamma \vdash e : \tau$ and $\Gamma \vdash s$ are a terminating algorithm (Prolog program!) for type checking.

3.2.2 Semantics for WHILE$_{\mathbf{T}}$

11/21/2001

$$\Sigma = Var \rightarrow Val, Val = T \cup \mathbb{Z}$$

where T is the set of truth values $\{$tt,ff$\}$

Assumption: T and $\mathbb{Z}$ are disjoint.
Need new value **err** $\notin Val$ to indicate a run time error.

$$Val_e = Val \cup \{\mathbf{err}\}$$

$\mathcal{A}[a_1\ aop\ a_2]\sigma = [aop](\mathcal{A}[a_1]\sigma, \mathcal{A}[a_2]\sigma)$

$$\begin{array}{rcl} [aop]_e : Val_e \times Val_e & \rightarrow & Val_e \\ [aop]_e(v_1, v_2) & = & \begin{cases} [aop](v_1, v_2) \text{ if } v_1, v_2 \in \mathbb{Z} \\ \mathbf{err} \text{ otherwise} \end{cases} \end{array}$$

Analogously:

$$[bop]_e : Val_e \times Val_e \rightarrow Val_e$$
$$[rop]_e : Val_e \times Val_e \rightarrow Val_e$$

$$\begin{array}{rcl} \mathcal{E} : Exp \rightarrow \Sigma & \rightarrow & Val_e \\ \mathcal{E}[n]\sigma & = & n \\ \mathcal{E}[x]\sigma & = & \sigma(x) \\ \mathcal{E}[\mathbf{true}\ /\mathbf{false}\ ]\sigma & = & \text{tt/ff} \\ \mathcal{E}[e_1\ aop\ e_2]\sigma & := & [aop]_e(\mathcal{E}[e_1]\sigma, \mathcal{E}[e_2]\sigma) \quad +\ bop,\ +\ rop \end{array}$$

Operational semantics for statements

Relation $Stm \times \Sigma \times \Sigma_e$ where $\Sigma_e = \Sigma \cup \{err\}$ $(err \notin \Sigma)$
Notation $\langle s, \sigma \rangle \rightarrow \sigma'$

Rules:

$$\begin{array}{rl} \langle \mathbf{skip}\ , \sigma \rangle \rightarrow \sigma & \\ \langle x := e, \sigma \rangle \rightarrow \sigma[x \mapsto \mathcal{E}[e]\sigma] & \text{if } \mathcal{E}[e]\sigma \neq \mathbf{err} \\ \langle x := e, \sigma \rangle \rightarrow \mathbf{err} & \text{if } \mathcal{E}[e]\sigma = \mathbf{err} \end{array}$$

Note: $I := 0$; $I := true$ is semantically ok, but not type correct!

$$\frac{\langle s_1, \sigma_1\rangle \rightarrow \sigma_2 \qquad \langle s_2, \sigma_2\rangle \rightarrow \sigma_3 \qquad (\sigma_2 \neq \textbf{err})}{\langle s_1; s_2, \sigma_1\rangle \rightarrow \sigma_3}$$

$$\frac{\langle s_1, \sigma_1\rangle \rightarrow \textbf{err}}{\langle s_1; s_2, \sigma_1\rangle \rightarrow \textbf{err}}$$

$$\frac{\mathcal{E}[e]\sigma \notin T}{\langle \textbf{if } e \textbf{ then } s_1 \textbf{ else } s_2, \sigma\rangle \rightarrow \textbf{err}}$$

$$\frac{\mathcal{E}[e]\sigma = \text{tt}/\text{ff} \qquad \langle s_1 \ / \ s_2, \sigma\rangle \rightarrow \sigma'}{\langle \textbf{if } e \textbf{ then } s_1 \textbf{ else } s_2, \sigma\rangle \rightarrow \sigma'}$$

$$\frac{\langle \textbf{if } e \textbf{ then } s; \textbf{while } e \textbf{ do } s \textbf{ else skip }, \sigma\rangle \rightarrow \sigma'}{\langle \textbf{while } e \textbf{ do } s, \sigma\rangle \rightarrow \sigma'}$$

Correctness of type system:
Type correct programs do not produce **err** (at runtime)

Completeness:
Programs that do not produce **err** are type correct.

$$(\textbf{if } b \textbf{ then } I := true \textbf{ else } I := 0; I := 1)$$

Correctness is desirable, completeness (in general) impossible.
Languages with a correct type system are called *type safe*:

- Examples: Java, ML, Haskell, Gofer, $C^\sharp$
- Counter-examples: Lisp, Eiffel, C, C++

3.2.3 WHILE$_T$ is type safe

Definition:

$$\begin{array}{rcl} typ & : & Val \rightarrow \{\textbf{int}, \textbf{bool}\} \\ typ(v) & = & \begin{cases} \textbf{int} & \text{if } v \in \mathbb{Z} \\ \textbf{bool} & \text{if } v \in T \end{cases} \end{array}$$

"σ is type correct with respect to Γ"
$\sigma : \Gamma \Leftrightarrow \forall x : t \in \Gamma. typ(\sigma(x)) = \tau$

Theorem: (Type safety)
$\Gamma \vdash s \wedge \sigma : \Gamma \wedge \langle s, \sigma\rangle \rightarrow \sigma' \Rightarrow \sigma' \neq \textbf{err} \wedge \sigma' : \Gamma$

Proof: by induction on $\langle s, \sigma\rangle \rightarrow \sigma'$

- Case $\langle x := e, \sigma\rangle \rightarrow \underbrace{\sigma[x \mapsto \mathcal{E}[e]\sigma]}_{\sigma'}, \ \mathcal{E}[e]\sigma \neq \textbf{err}$

$$\Gamma \vdash x := e \quad \Rightarrow \quad x : \tau \in \Gamma \wedge \Gamma \vdash e : \tau$$

$$\Rightarrow \quad typ(\mathcal{E}[e]\sigma) = t \ \text{(because } \sigma : \Gamma)$$

Lemma: $\Gamma \vdash e : \tau \wedge \sigma : \Gamma \Rightarrow typ(\mathcal{E}[e]\sigma) = \tau$

$$\begin{aligned} \sigma : \Gamma \quad &\Rightarrow \quad typ(\sigma(x)) = \tau = typ\left(\sigma'(x)\right) \\ &\Rightarrow \quad \sigma' : \Gamma \end{aligned}$$

3.3 A polymorphic type system

11/26/2001

Keywords: type variables, genericity

3.3.1 Type checking

Example: $x : \tau;\ y : \tau;\ x := y$ is type correct for any type τ

New syntax for types:

$$\tau ::= \textbf{int} \mid \textbf{bool} \mid \underbrace{\alpha \mid \beta \mid \gamma \mid \ldots}_{\text{type variable}}$$

Example: $x : \alpha;\ y : \alpha;\ x := y$

→ Semantics is unchanged because it is independent of the type language.

→ Type checking rules do not change either! Why? Because for type checking, type variables are treated like new atomic types.

$$\frac{\dfrac{x : \alpha \in \Gamma}{\Gamma \vdash x : \alpha} \qquad \dfrac{y : \alpha \in \Gamma}{\Gamma \vdash y : \alpha}}{\underbrace{x : \alpha;\ y : \alpha}_{\Gamma} \vdash x := y}$$

Problem with type safety theorem: $\sigma : \Gamma$ only holds if Γ is monomorphic (does not contain type variables).

Idea: A (type variable) *substitution* is a function

$$\Theta : \ \text{type variables} \ \rightarrow \ \text{types}$$

Example:

$$\Theta = \{a \mapsto \textbf{bool}\,,\ \beta \mapsto \textbf{int}\,,\ \gamma \mapsto \delta, \ldots\}$$

Extension to types:

$$\begin{aligned} \Theta(\textbf{int}) \quad &= \quad \textbf{int} \\ \Theta(\textbf{bool}) \quad &= \quad \textbf{bool} \end{aligned}$$

Extension to contexts:

$$\Theta\Big(\{x_1 : \tau_1; \ldots\}\Big) = \{x : \Theta(\tau_1); \ldots\}$$

Theorem: $\Gamma \vdash s \wedge \sigma : \Theta(\Gamma) \wedge \langle s, \sigma\rangle \to \sigma' \Rightarrow \sigma' \neq \textbf{err} \wedge \sigma' : \Theta(\Gamma)$

Proof: similar to monomorphic case.
"If s is type correct with respect to Γ, then type safe execution is guaranteed for any state σ that is type correct with respect to an instance $\Theta(\Gamma)$ of Γ."

Example:

$$\{x : \alpha, y : \alpha\} \vdash s \Rightarrow \text{ start state } \begin{cases} \{x \mapsto 0, y \mapsto 1\} & \text{ok} \\ \{x \mapsto 0, y \mapsto \text{tt}\} & \text{not ok} \\ \{x \mapsto \text{tt}, y \mapsto \text{tt}\} & \text{ok} \end{cases}$$

3.3.2 Type inference / reconstruction

Compute missing declarations **Example:**

$$\begin{array}{rcl} \textbf{while } I > 0 \textbf{ do } I := I + 1 & \Rightarrow & I : \textbf{int} \\ \textbf{while } I > 0 \textbf{ do } I := I \wedge I & & \text{:-(} \\ x := y & \Rightarrow & x : \tau, y : \tau \text{ for any } \tau \end{array}$$

Key ideas:

1. Use type variable during reconstruction to stand for (yet) unknown types.
2. Use type checking rules for type reconstruction by interpreting the rules as a Prolog program, i.e. use rules backwards and instantiate type variables by unification.

$$\cfrac{\cfrac{\cfrac{(z:\gamma) \in \Gamma \quad \overset{\Rightarrow \beta=\gamma}{\Gamma \vdash y : \gamma}}{\Gamma \vdash z := y} \qquad \cfrac{(y:\beta) \in \Gamma \quad \overset{\Rightarrow \alpha=\beta}{\Gamma \vdash x : \beta}}{\Gamma \vdash y := x}}{\Gamma \vdash z := y;\ y := x} \qquad \cfrac{(x:\alpha) \in \Gamma \quad \overset{\Rightarrow \alpha=\textbf{int}}{\Gamma \vdash 0 : \alpha}}{\Gamma \vdash x := 0}}{\underbrace{x : \alpha, y : \beta, z : \gamma}_{\Gamma} \vdash (z := y;\ y := x);\ x := 0}$$

In Prolog:

$$\begin{array}{rcl} \text{Query} & : & \text{typecheck } ([x : \alpha, y : \beta, z : \gamma], z := y;\ y := x;\ x := 0) \\ \text{Result} & : & \alpha = \textbf{int} \\ & & \beta = \textbf{int} \\ & & \gamma = \textbf{int} \end{array}$$

Algorithm for type inference: Beginning with

$$\underbrace{\{x_1 : \alpha_1, \ldots, x_n : \alpha_n\}}_{\Gamma} \vdash s$$

apply the rules for $\Gamma \vdash s$ backwards while instantiating the α_i as much as necessary (unification!) $\Rightarrow$ We compute a substitution Θ such that $\{x_1 : \Theta(\alpha_1), \ldots, x_n : \Theta(\alpha_n)\} \vdash s$ is provable.

⇒ Type inference reduced to type checking

⇒ Type safety also guaranteed for type inference

Theorem: Let Θ be computed by the above algorithm. Then

1. $\Theta(\Gamma) \vdash s$
2. Θ is the "most general solution"
   If $\Theta'(\Gamma) \vdash s$ is provable, then Θ is more general than Θ' (Θ' is an *(*instance) of Θ):
   $\Theta' = \delta \circ \Theta$ for some substitution δ.

Example: For $x := y$, $\Gamma = \{x : \alpha, y : \beta\}$ the algorithm computes $\Theta = \{\beta \mapsto \alpha\}$
$\Theta' = \{\alpha \mapsto$ **bool** $, \beta \mapsto$ **bool** $\}$ is an instance of Θ: $\Theta' = \{\alpha \mapsto$ **bool** $\} \circ \Theta$

Classification of programming languages with respect to type systems

	monomorphic (no reuse)	**polymorphic**[7]
static	Pascal, Modula, (Fortran)	ML, Haskell, Gofer, C, Java, $C^\sharp$
dynamic	not sensible	LISP

[7] allows generic (with respect to types) programs

4 Denotational Semantics

11/28/2001

1. The semantics of inductive definitions (including rule induction)
2. A denotational semantics of While based on sets
3. Fixpoint theory
4. A direct denotational semantics of While
5. Extensions of While

4.1 Inductive Definitons

Rules:

$$\frac{a_1 \in X \qquad \dots \qquad a_n \in X}{a \in X}$$

defines $X \subseteq A$ where X is the set to be defined and $a_1, \dots, a_n, a \in A$. In the sequel: $R \subseteq \wp(A) \times A$ is a rule.

Note: R, X is potentially infinite.

Definition: $B \subseteq A$ is *R-closed* iff. for all $(H, a) \in R$:

$$H \subseteq B \Rightarrow a \in B$$

Definition: I_R is the least R-closed subset of A.

Note: There is always a least R-closed subset.

$$B_1, B_2 \text{ R-closed } \Rightarrow B_1 \cap B_2 \text{ R-closed}$$

$I_R = \bigcap\{B \subseteq A \mid \text{ B is R-closed } \}$

Why the least set? No junk! I_R should contain only those elements that R forces to be in it.

Example: $A = \mathbb{R}$, Rules: $\frac{}{0 \in X}$, $\frac{n \in X}{n+1 \in X}$

Least R-closed subset of $\mathbb{R}$: $\mathbb{N}$. Also R-closed: $\mathbb{R}, \mathbb{Q}, \mathbb{Z}$

Set theoretic way of writing $\mathbb{R}$:

$$\{(\emptyset, 0)\} \cup \{(\{n\}, n+1) \mid n \in \mathbb{R}\}$$

4.1.1 Rule induction

Aim:

$$\frac{\forall (\{h_1, \dots, h_n\}, a) \in R . P(h_1) \wedge \dots \wedge P(h_n) \Rightarrow P(a)}{\forall x \in I_R . P(x)}$$

$\langle s, \sigma \rangle \rightarrow \sigma'$
$(s, \sigma, \sigma') \in OpSem$

$$I_R \subseteq P \Rightarrow I_R = I_R \cap P \Rightarrow I_R \cap P \text{ is R-closed} \overset{\text{leastness of } I_R}{\Longrightarrow} I_R \subseteq I_R \cap P \Rightarrow I_R \subseteq P$$

$$\forall x \in I_R.P(x)) \quad \Leftrightarrow \quad (\forall (H,a) \in R.\underbrace{H \subseteq I_R}_{\text{additional assumption}} \wedge (\forall h \in H.P(h)) \Rightarrow \underbrace{a \in I_R}_{\text{tt}} \wedge P(a))$$

$$\overset{\text{Rule induction}}{\Leftarrow}$$

Generalized rule induction:

$$\frac{\forall (H,a) \in R.H \subseteq I_R \wedge (\forall h \in H.P(h)) \Rightarrow P(a)}{\forall x \in I_R.P(x)}$$

Side conditions

Example:

$$\frac{B[b]\sigma = \text{ff}}{\langle w, \sigma \rangle \rightarrow \sigma'}$$

Need to distinguish two kinds of premises:

a_i and everything else ("side conditions")

$$\frac{a_1 \in X, \ldots, a_n \in X \qquad C_1 \ldots C_m}{a \in X} \tag{4}$$

With side conditions:

$$\frac{\forall \text{ rules } (4)\ .a_1, \ldots, a_n \in I_R \wedge P(a_1) \wedge \ldots \wedge P(a_n) \wedge C_1 \wedge \ldots \wedge C_m \Rightarrow P(a)}{\forall x \in I_R.P(x)}$$

4.1.2 I_R as a least fixed point ($f(x) = x$)

Assumption: R is *finitary*: $(H,a) \in R \Rightarrow H$ finite

$\hat{R} : \wp(A) \rightarrow \wp(A)$
$\hat{R}(B) = \{a \mid \exists (H,a) \in R.H \subseteq B\}$ "one step consequence operator"

To show:

1. I_R is least fixpoint of $\hat{R}$
2. $I_R = \bigcup_{i \in \mathbb{N}} \hat{R}^i(\emptyset)$

Easy to see:

1. B is R-closed iff. $\hat{R}(B) \subseteq B$
2. $\hat{R}$ is monotone ($B_1 \subseteq B_2 \Rightarrow \hat{R}(B_1) \subseteq \hat{R}(B_2)$)

$\hat{R}$ generates/approximates I_R:

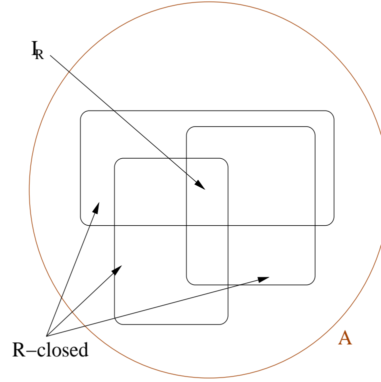
$$\begin{aligned} A_0 &:= \hat{R}^0(\emptyset) = \emptyset \text{ - at the start we know nothing} \\ A_1 &:= \hat{R}^1(\emptyset) = \hat{R}(\emptyset) \text{ - all axioms/facts} \\ A_{n+1} &:= \hat{R}^{n+1}(\emptyset) = \hat{R}(\hat{R}^n(\emptyset)) = \hat{R}(A_n) \\ &\quad \text{- all elements of } A \text{ that can be derived with a derivation tree of depth } n+1 \end{aligned}$$

$A_\omega := \bigcup_{i\in\mathbb{N}} A_i$

Example: $A = \mathbb{N}$ Rules: $\frac{}{0 \in X}$, $\frac{n \in X}{n+2 \in X}$

$$\begin{array}{rcl} \hat{R}(M) & = & \{0\} \cup \{m+2 \mid m \in M\} \\ \hat{R}^0(\emptyset) & = & \emptyset \\ \hat{R}^1(\emptyset) & = & \{0\} \\ \hat{R}^2(\emptyset) & = & \{0,2\} \\ \hat{R}^3(\emptyset) & = & \{0,2,4\} \\ \hat{R}^n(\emptyset) & = & \{0,\ldots,2(n-1)\} \end{array}$$

$\bigcup_{i\in\mathbb{N}} \hat{R}^i(\emptyset) =$ set of even numbers



The $(A_i)_{i\in\omega}$ forms an ω-chain:

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq \ldots \subseteq A_n \subseteq \ldots$$

Proof: $A_i \subseteq A_{i+1}$ by induction on i

$$\begin{array}{rcl} i=0 & : & A_0 = \emptyset \subseteq A_1 \\ i \to i+1 & : & A_i \subseteq A_{i+1} \stackrel{?}{\Rightarrow} A_{i+1} \subseteq A_{i+2} \\ & & \hat{R} \text{ monotone: } A \subseteq B \Rightarrow \hat{R}(A) \subseteq \hat{R}(B) \\ & & \hat{R}(A_i) \subseteq \hat{R}(A_{i+1}) \end{array}$$

Theorem: If R is finitary

- A_ω is R-closed
- $\hat{R}(A_\omega) = A_\omega$
- $A_\omega = I_R$

Proof:

- Let $(\{h_1,\ldots,h_n\},a) \in R$, $\{h_1,\ldots,h_n\} \subseteq A_\omega$
  Show $a \in A_\omega$:
  $\forall 1 \leq i \leq n. \exists k_i. h_i \in A_{k_i}$
  $k := max\{k_1,\ldots,k_n\} \Rightarrow \forall 1 \leq i \leq n. k_i \in A_k \Rightarrow a \in A_{k+1} \Rightarrow a \in A_\omega$

- $\hat{R}(A_\omega) \subseteq A_\omega \wedge A_\omega \subseteq \hat{R}(A_\omega)$

  Let $a \in A_\omega$; show $a \in \hat{R}(A_\omega)$
  $a \in A_\omega \Rightarrow \exists n.a \in A_n \Rightarrow a \in \hat{R}(A_n) \subseteq \hat{R}(A_\omega)$
  because $A_n \subseteq A_\omega$ and $\hat{R}$ monotone

- Show A_ω is the least R-closed set. Let B be an R-closed set ($\hat{R}(B) \subseteq B$).
  Show $A_\omega \subseteq B : \forall i.A_i \subseteq B$ by induction on i.

$$\begin{aligned} i = 0 \quad &: \quad A_0 \subseteq B \ (A_0 = \emptyset) \\ i \to i+1 \quad &: \quad A_i \subseteq B \overset{?}{\Rightarrow} A_{i+1} \subseteq B = \hat{R}(A_i) \subseteq \hat{R}(B) \end{aligned}$$

Corollary:

1. $I_R = \bigcup_{i \in \omega} \hat{R}^i(\emptyset)$
2. I_R is least fixpoint of $\hat{R}$

Proof: for 2:
Show: every fixpoint of $\hat{R}$ is R-closed:

$$\hat{R}(B) = B \Rightarrow \hat{R}(B) \subseteq B \Rightarrow B \text{ is R-closed}$$

Notation: $A_\omega = fix(\hat{R})$

4.2 Denotational Semantics of WHILE

- operational: close to implementation
- denotational: more abstract, easier to use in proofs

Idea: For every syntactic class C (Stmt, Aexp, ...) there is one function $D_C : C \to M$, where M is a semantic domain, e.g. a known mathematical structure.

- $A : Aexp \to (\Sigma \to \mathbb{N})$
- $S : Stmt \to (\Sigma \to \Sigma)$ (operation sem.: $Stmt \times \Sigma \times \Sigma$)
  - $S[skip] = Id$ $\quad$ ($S[skip]\sigma = \sigma$)
  - $S[x := a]\sigma = \sigma[x \mapsto A[a]\sigma]$
  - $S[s_1; s_2] = S[s_2] \circ S[s_1]$ $\quad$ ($S[s_1; s_2]\sigma = S[s_2](S[s_1]\sigma)$)
  - S[if b then s_1 else s_2]$\sigma =$
    - $S[s_1]\sigma$ if $B[b]\sigma = $ tt
    - $S[s_2]\sigma$ if $B[b]\sigma = $ ff

– a try for while:

$$S[\text{while b do s}]\sigma = \begin{cases} \sigma & \text{if } B[b] = \text{ff} \\ S[w](S[s]\sigma) & \text{if } B[b] = \text{tt} \end{cases} \tag{5}$$

Ω = while true do skip
$S[\Omega] = S[\Omega] \circ S[skip]$[8] $= S[\Omega]$:-(

Wanted: least function satisfying (5) (here: "least" means least defined function)
Why? Termination only if explicitly required by the definition ("no garbage").
Equivalence to operational semantics.

General principle for recursive definitions / solutions to recursive equations

Recursive:

$$f = \underbrace{\ldots f \ldots}_{e} \quad f \in A \text{ (e.g. } A = B \to C) \tag{6}$$

Wanted: *Least* solution f. Define $F(f) := e$ [9]
Now if f is the solution to (6) $\Rightarrow F(f) = f$

Wanted: $fix(F) =: f$, then f is by definition the least solution of (6).

Problem: When does $fix(F)$ exist?
First solution for while: define $\Sigma \to \Sigma$ as set

1. Transform $S[w]\sigma$ into a recursive equation for $S[w]$, treating functions ($S[w]\sigma$) as relations ($S[w]$).

$$S[w] = \{(\sigma,\sigma) \mid B[b]\sigma = \text{ff}\} \cup \underbrace{\{(\sigma, S[w](S[s]\sigma)) \mid B[b]\sigma = \text{tt} \wedge \exists \sigma'.(\sigma,\sigma') \in S[s] \wedge (\sigma',\sigma'') \in S[w]\}}_{\Gamma(S[w])}$$

Wanted: $fix(\Gamma)$
$\Gamma : (\Sigma \nrightarrow \Sigma) \to (\Sigma \nrightarrow \Sigma)$
Intuition: if $\Phi : \Sigma \nrightarrow \Sigma$ is the meaning of some $s' \in Stm$ then $\Gamma(\Phi)$ is the meaning of **if** b **then** $s;s'$ **else skip**
$\Rightarrow \Gamma$ unfolds the loop once.
if b **then** $\{s$; **if** b **then** $s;\ldots$ **else skip** $\}$ **else skip** $\}$ infinite unfolding

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2. Find set of rules R such that $\hat{R} = \Gamma$.

$$R = \{(\emptyset, (\sigma,\sigma)) \mid B[b]\sigma = \text{ff}\} \cup \{(\{(\sigma',\sigma'')\}, (\sigma,\sigma'')) \mid B[b]\sigma = \text{tt} \wedge (\sigma,\sigma') \in S[s]\}$$

Fact: $\hat{R} = \Gamma$

Definition: $S[w] = fix(\Gamma) = \bigcup_{i\in\mathbb{N}} \hat{R}^i(\emptyset)$ $[N.B.\ S[w] \in \Sigma \nrightarrow \Sigma]$

Intuition: $\hat{R}^i(\emptyset)$ = semantics of the first i iterations

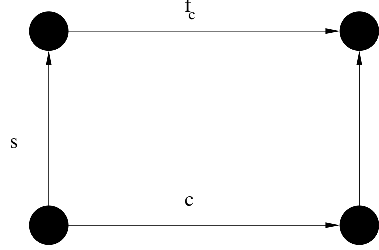
Denotational semantics is *compositional*:

$$S[c(t_1,\ldots,t_n)] = f_c(S[t_1],\ldots,S[t_n])$$

c is composition of statements

[8] Id
[9] $F \in A \to A$

Recursion theory: primitive recursion
Mathematics: Homomorphism (from syntax to semantics)



Theorem: $S[w] = S[$**if** b **then** $s; w$ **else skip** $]$

Proof:

$$\begin{array}{rcl} S[w]\sigma & = & fix(\Gamma)\sigma = \Gamma(fix(\Gamma))\sigma = \Gamma(S[w])\sigma \\ & = & \begin{cases} \sigma \;\; if B[b]\sigma = \text{ff} \\ S[w](\underbrace{S[s]\sigma}_{\sigma'}) \;\; if B[b]\sigma = \text{tt} \end{cases} \\ & = & \begin{cases} S[\textbf{skip}]\sigma \;\; if B[b]\sigma = \text{ff} \\ S[s;w]\sigma \;\; if\ B[b]\sigma = \text{tt} \end{cases} \\ & = & S[\textbf{if}\ \text{b}\ \textbf{then}\ \text{s;w}\ \textbf{else skip}]\sigma \end{array}$$

Equivalence of operational & denotational semantics

Aim: $\langle s, \sigma \rangle \to \sigma' \Leftrightarrow S[s]\sigma = \sigma'$
Theorem: $\langle s, \sigma \rangle \to \sigma' \Rightarrow S[s]\sigma = \sigma'$

Proof: by induction on $\langle s, \sigma \rangle \to \sigma'$

$$\frac{\langle s, \sigma \rangle \to \sigma' \quad \langle w, \sigma' \rangle \to \sigma''}{\langle w, \sigma \rangle \to \sigma''} B[b]\sigma = \text{tt}$$

$$\begin{array}{rcl} S[w]\sigma & = & \Gamma(S[w])\sigma \\ & \stackrel{\Gamma}{=} & S[w](S[s]\sigma) \\ & \stackrel{I.H.1}{=} & S[w]\sigma' \\ & \stackrel{I.H.2}{=} & \sigma'' \\ & & \text{(Leastness not required)} \end{array}$$

Theorem: $S[s]\sigma = \sigma' \Rightarrow \langle s, \sigma \rangle \to \sigma'$

Proof: by induction on s

- Case $s = s_1; s_2$
  $S[s]\sigma = S[s_2](S[s_1]\sigma)$

$\Rightarrow \exists \sigma_1 . S[s_1]\sigma = \sigma_1$
$\overset{I.H.}{\Rightarrow} \langle s_1, \sigma \rangle \rightarrow \sigma_1$
$\overset{I.H.}{\Rightarrow} \langle s_2, \sigma \rangle \rightarrow \sigma'$
$\Rightarrow \langle s_1; s_2, \sigma \rangle \rightarrow \sigma'$

- $s = \textbf{while } b \textbf{ do } s_0$
  $S[s]\sigma = \sigma'$
  $\Rightarrow (\sigma, \sigma') \in S[s] = fix(\Gamma) = \bigcup_{n \in \mathbb{N}} \hat{R}^n(\emptyset)$
  $\Rightarrow \exists n . (\sigma, \sigma') \in \hat{R}^n(\emptyset)$
  $\Rightarrow \langle s, \sigma \rangle \rightarrow \sigma'$
  by auxiliary lemma
  $R = \{(\emptyset, (\sigma, \sigma)) \mid B[b]\sigma = \text{ff}\} \cup \{(\{(\sigma', \sigma'')\}, (\sigma, \sigma'')) \mid B[b]\sigma = \text{tt} \wedge (\sigma, \sigma') \in S[s_0]\}$

  Lemma: (local)
  $\forall \sigma, \sigma' . (\sigma, \sigma') \in \hat{R}^n(\emptyset) \Rightarrow \langle s, \sigma \rangle \rightarrow \sigma'$

  Proof: by induction on n

  - $n = 0 : \checkmark$
  - $n \rightarrow n+1 : (\sigma, \sigma') \in \hat{R}^{n+1}(\emptyset) = \hat{R}(\hat{R}^n(\emptyset))$
    Case distinction:
    * $B[b]\sigma = \text{ff}\checkmark$
    * $B[b]\sigma = \text{tt} : \exists \sigma'' . (\sigma, \sigma'') \in S[s_0] \wedge (\sigma'', \sigma') \in \hat{R}^n(\emptyset)$
      $\Rightarrow \underbrace{\langle s_0, \sigma \rangle \rightarrow \sigma''}_{\text{induction on s}} \wedge \langle s, \sigma'' \rangle \rightarrow \sigma'$
      $\Rightarrow \langle s, \sigma \rangle \rightarrow \sigma'$ qed.

Theorem:

$$\langle s, \sigma \rangle \rightarrow \sigma' \Leftrightarrow S[s]\sigma = \sigma'$$

4.3 Complete partial orders (cpos)[10]

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Generalization of $\wp A, \hat{R} : \wp A \rightarrow \wp A \quad fix(\hat{R}) = \bigcup \ldots$
Definition: Let $(P, \sqsubseteq)$ be a partial order, $X \subseteq P, u \in P$.

- u is an *upper bound* of X iff. $\forall x \in X . x \sqsubseteq u$
- u is *least upper bound*[11] of X iff. u is an upper bound of X and if u' is an upper bound of X then $u \sqsubseteq u'$

[10] "Vollständige Halbordnungen"
[11] least upper bound = "Supremum"

Remark: Least upper bounds don't need to exist ($\underbrace{..}_{X}$), but if they exist, they are unique: u_1, u_2 are least upper bounds of X then $u_2 \sqsubseteq u_1$ and $u_1 \sqsubseteq u_2$ and therefore $u_1 = u_2$.

Notation: If X has a least upper bound, it is written $\bigsqcup X$
$x_1 \sqcup x_2 := \bigsqcup\{x_1, x_2\}$. NB: $x \sqsubseteq y \Rightarrow x \sqcup y = y$

Definition: Let $(D, \sqsubseteq)$ be a partial order (D = "Domain"). $(D, \sqsubseteq)$ is a *complete partial order* (cpo) iff. all ω-chains $K \subseteq D$ have a least upper bound $\bigsqcup K$.
An ω-chain is a set $\{d_0, d_1, \ldots\}$ such that $d_0 \sqsubseteq d_1 \sqsubseteq \ldots$
$(D, \sqsubseteq)$ is a *cpo with* $\perp$ if $\perp \in D$ is the least element of D.

Example:
$\left(P(A), \subseteq\right)$ is a cpo with $\perp = \emptyset$ and $\bigsqcup = \bigcup$

$(A \nrightarrow A, \sqsubseteq)$
$f := (\bigcup_{i\in\mathbb{N}} f_i) \in A \nrightarrow A$ if each $f_i \in A \nrightarrow A$ and if $f_0 \sqsubseteq f_1 \sqsubseteq \ldots$

Proof: $(a_1, a_2), (a_1, a_3) \in f$
Without loss of generality $(a_1, a_2) \in f_i, (a_1, a_3) \in f_j, i \le j$
$\Rightarrow f_i \subseteq f_j \Rightarrow (a_1, a_2) \in f_j \in A \nrightarrow A$
$\Rightarrow a_2 = a_3$

Generalization:

$$\begin{array}{rcl} \wp A & \rightsquigarrow & D \\ \subseteq & \rightsquigarrow & \sqsubseteq \\ \emptyset & \rightsquigarrow & \perp \\ \hat{R} & \rightsquigarrow & \text{continuous function} \end{array}$$

Why does $\hat{R}$ have a least fixpoint? Because it is *monotone*.
Why $fix(\hat{R}) = \bigcup_{i\in\mathbb{N}} \hat{R}^i(\emptyset)$? Because $\hat{R}$ is *continuous*: $\hat{R}(\bigcup_{i\in\mathbb{N}} B_i) = \bigcup_{i\in\mathbb{N}} \hat{R}(B_i)$ where $B_0 \subseteq B_1 \subseteq \ldots$

Counter-example to continuity:

$$\frac{}{0} \quad \frac{n}{n+2} \quad \frac{0 \quad 2 \quad 4 \quad \ldots}{1}$$

$B_i = \{0, 2, 4, \ldots, 2(i-1)\}$
$B_0 = \emptyset, B_1 = \{0\}, B_2 = \{0, 2\}, \ldots$

$$\begin{array}{c} \hat{R}(\bigcup_i B_i) = \hat{R}(\{0,2,4,\ldots\}) = \{0,2,4,\ldots\} \cup \{1\} \\ \neq \\ \bigcup_i \hat{R}(B_i) = \bigcup_i B_{i+1} = \{0,2,4,\ldots\} \end{array}$$

Definition:
Let $(D, \sqsubseteq_D)$ and $(E, \sqsubseteq_E)$ be cpos, and let $f : D \rightarrow E$.
f is monotone iff. $d_1 \sqsubseteq_D d_2 \Rightarrow f(d_1) \sqsubseteq_E f(d_2)$
f is continuous iff. f is monotone and for all ω-chains $d_0 \sqsubseteq d_1 \sqsubseteq \ldots$ we have

$$\underbrace{f(\bigsqcup_i d_i)}_{\text{exists because } D \text{ cpo}} = \underbrace{\bigsqcup_i f(d_i)}_{\text{exists because } E \text{ cpo}}$$

Theorem: (Knaster-Tarski Fixpoint theorem)
Let $(D, \sqsubseteq)$ be a cpo with $\bot$ and $f : D \to D$ be continuous[12]. Then $fix(f) = \bigsqcup f^i(\bot)$ is the least fixpoint of f.

Proof:
$\bigsqcup_i f^i(\bot)$ exists because $\{f^i(\bot)\}_{i\in\mathbb{N}}$ is an ω-chain:
(Proof by induction on $i : f^i(\bot) \sqsubseteq f^{i+1}(\bot)$)

$\bot \sqsubseteq f(\bot) \sqsubseteq f(f(\bot))\dots$

(i) $f(fix(f)) = fix(f)$:

$$\begin{aligned} f(\bigsqcup_i f^i(\bot)) &= \bigsqcup_i f^{i+1}(\bot) \\ &= \bot \sqcup \bigsqcup_i f^{i+1}(\bot) \\ &= f^0(\bot) \sqcup \bigsqcup_i f^{i+1}(\bot) = \bigsqcup_i f^i(\bot) \\ &\overset{def}{=} fix(f) \end{aligned}$$

$(f^0(x) = x)$

(ii) $f(p) \sqsubseteq p \Rightarrow fix(f) \sqsubseteq p$
By induction on $i : f^i(\bot) \sqsubseteq p$

(a) $f^0(\bot) = \bot \sqsubseteq p$

(b) $f^{i+1}(\bot) = \underbrace{f(f^i(\bot)) \sqsubseteq f(p)}_{\text{I.H. + mono}(f)} \sqsubseteq p$

$\Rightarrow \bigsqcup_i f^i(\bot) \sqsubseteq p$ because p is an upper bound of $\{f^i(\bot)\}_{i\in\mathbb{N}}$ and $\bigsqcup$ is the least upper bound.

Dana Scott: "Domain theory"

Thesis: Any computable function is continuous with respect to a suitable ordering.

$$f(x) = \textbf{if } x \leq 0 \textbf{ then } 0 \textbf{ else } 1$$

not continuous in the sense of analysis.

Application to **WHILE** :

1. $\Sigma \rightharpoonup \Sigma$ is a cpo
2. For **while** : $\Gamma : (\Sigma \rightharpoonup \Sigma) \to (\Sigma \rightharpoonup \Sigma)$ has a least fixpoint because:
   a) $\Gamma = \hat{R}$ for a finitary R. ✓
   b) Γ is continuous – tedious[13] if proved directly. Better: build general theory of continuous functions, show that Γ is a composition of continuous functions.

[12] "stetig"
[13] "mühsam", "langweilig, "nervtötend"

λ-Notation

$$\begin{array}{rcl} f(x) = x+1 & \leadsto & f = \lambda x.x+1 \\ f(5) = 5+1 & \leadsto & f\ 5 = (\lambda x.x+1)5 \rightarrow (x+1)[5/x] = 5+1 \end{array}$$

$\lambda x.t$: x parameter, t result

Typing rules:

$$\frac{x : D \qquad t : E}{\lambda x.t : D \rightarrow E} \qquad \frac{f : D \rightarrow E \qquad t : D}{f\ t : E}$$

α-Conversion[14]: $\lambda x.x+1 = \lambda y.y+1$

$$\begin{array}{rcl} \lambda x.x+y & \neq & \lambda x.x+z \\ \lambda x.\lambda y.x+y & \neq & \lambda x.\lambda x.x+x \end{array}$$

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$$\beta : (\lambda x.s)t \quad =_\beta \quad s[t/x] \text{ (Substitution of } t \text{ for } x \text{ in } s\text{)}$$

$$\begin{array}{rcl} (\lambda x.\lambda y.x+y)y & \neq & \lambda y.y+y \\ = (\lambda x.\lambda y'.x+y')y & & \\ & = & \lambda y'.y+y' \end{array}$$

⇒ Definition of $s[t/x]$ needs to rename bound variables to avoid name clashes.

$$\eta : \lambda x.(t\ x) =_\eta t \quad \text{if } x \text{ does not occur free in } t$$

Example:

$$\begin{array}{rcl} \lambda x.\sin x & = & \sin \\ \lambda x.(x\ x) & \neq & x \end{array}$$

Construction of cpos

Aim: meta-language for defining semantics.

4.3.1 Discrete cpos

Definition: $\sqsubseteq$ is a *discrete* partial order iff. $\forall x,y.x \sqsubseteq y \Leftrightarrow x = y$

Fact: Every discrete partial order is a cpo.

Fact: If D is discrete, and $f : D \rightarrow E$, then f is continuous.

[14] "Gebundene Namen sind Schall und Rauch."

4.3.2 Cartesian Products

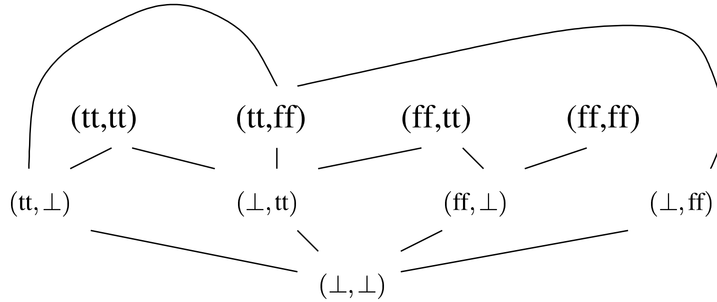
$D := D_1 \times \ldots \times D_n$ (D_i are cpos)
$(d_1, \ldots, d_n) \sqsubseteq_D (d'_1, \ldots, d'_n) \Leftrightarrow_{def} (d_1 \sqsubseteq_{D_1} d'_1) \wedge \ldots \wedge (d'_n \sqsubseteq_{D_n} d'_n)$
$(D, \sqsubseteq_D)$ is cpo:

$$\bigsqcup_{i \in \mathbb{N}} (d_1^i, \ldots, d_n^i) := (\bigsqcup_{i \in \mathbb{N}} d_1^i, \ldots, \bigsqcup_{i \in \mathbb{N}} d_n^i) \tag{7}$$

Need to check that right hand side of (7) exists and is least upper bound of left hand side.

Example:
$T_\perp := \frac{\text{tt} \quad \text{ff}}{\perp}$ (means: $\perp \sqsubseteq$ tt, $\perp \sqsubseteq$ ff)

$T_\perp \times T_\perp$:



The empty Cartesian product $\mathbf{1} = \{()\}$????

Lemma: Let E be a cpo, $e_{mn} \in E$ for $m, n \in \mathbb{N}$ such that $m \leq m' \wedge n \leq n' \Rightarrow e_{mn} \sqsubseteq_E e_{m'n'}$.
Then $\bigsqcup_i \bigsqcup_j e_{ij} = \bigsqcup_{i,j} e_{ij} = \bigsqcup_n e_{nn}$

$$\begin{array}{ccccc} \vdots & & & \iddots & \\ e_{01} & \sqsubseteq & e_{11} & \sqsubseteq & \ldots \\ & \rotatebox{45}{$\sqsubseteq$} & & & \\ e_{00} & \sqsubseteq & e_{10} & \sqsubseteq & \ldots \end{array}$$

Theorem: $f : \underbrace{D_1 \times \ldots \times D_n}_{D} \to E$ is continuous iff. it is continuous in each argument:
for all $1 \leq i \leq n$ and all $(d_1, \ldots, d_n) \in D$

$$f_i(x) := f(d_1, \ldots, d_{i-1}, x, d_{i+1}, \ldots, d_n) \text{ is continuous}$$

Proof:
$\Rightarrow \checkmark$
$\Leftarrow$: (only case $n = 2$)

1. f is monotone: $(x_0, y_0) \sqsubseteq (x_1, y_1) \Leftrightarrow$

$$f(x_0, y_0) \overset{f_1 \text{ monotone}}{\sqsubseteq} f(x_1, y_0) \overset{f_2 \text{ monotone}}{\sqsubseteq} f(x_1, y_1)$$

2. $(x_0, y_0) \sqsubseteq (x_1, y_1) \sqsubseteq \dots$

$$\begin{aligned} f\left(\bigsqcup_i (x_i, y_i)\right) &= f\left(\bigsqcup_i x_i, \bigsqcup_j y_j\right) \\ &= \bigsqcup_i f\left(x_i, \bigsqcup_j y_j\right) \quad f_1 \text{ is continuous} \\ &= \bigsqcup_i \bigsqcup_j f(x_i, y_j) \quad f_2 \text{ is continuous} \\ &= \bigsqcup_i f(x_i, y_i) \quad \text{qed.} \end{aligned}$$

4.3.3 Function spaces

Let D, E be cpos.
$[D \to E] :=$ set of continuous functions from D to E.
$f \sqsubseteq_{[D \to E]} g \Leftrightarrow_{def} \forall d \in D. f(d) \sqsubseteq_E g(d)$ "pointwise ordering" **Fact:** $\sqsubseteq_{[D \to E]}$ is a cpo.

If $\perp$ is least element of E, then $\lambda d.\perp$ is least element of $[D \to E]$.
Let $f_0 \sqsubseteq f_1 \sqsubseteq \dots$ be an ω-chain in $[D \to E]$.

$$\left(\begin{aligned} &\Rightarrow \quad f_0(d) \sqsubseteq f_1(d) \sqsubseteq \dots \\ &\Rightarrow \quad \bigsqcup_i f_i(d) \text{ exists} \end{aligned} \right)$$

Then the f_i have a supremum:

$$\bigsqcup_i f_i = \lambda d. \bigsqcup_i f_i(d)$$

Need to check that right hand side is supremum and right hand side is continuous.

4.3.4 Lifting

$$D \leadsto D_\perp$$

Let D be a set. $D_\perp := \{\perp\} \cup \{\lfloor d \rfloor \mid d \in D\}$ (disjoint union)
$\lfloor . \rfloor : D \to D_\perp$ "Boxing / Tagging"
such that $\lfloor d \rfloor \neq \perp$
$\lfloor d_1 \rfloor = \lfloor d_2 \rfloor \Rightarrow d_1 = d_2$

Theorem: $\lfloor . \rfloor$ always exists ("by construction").

$T_\perp = \{\lfloor \text{tt} \rfloor, \lfloor \text{ff} \rfloor, \perp\}$

If D is a (c)po then $D_\perp$ is a (c)po:

$$d_1' \sqsubseteq_{D_\perp} d_2' \Leftrightarrow_{def} (d_1' = \perp) \vee \exists d_1, d_2 \in D. d_1' = \lfloor d_1 \rfloor \wedge d_2' = \lfloor d_2 \rfloor \wedge d_1 \sqsubseteq_D d_2$$

If D is discrete, then $D_\perp$ is called a *flat* cpo. $x \sqsubseteq y \Leftrightarrow x = \perp \vee x = y$

$$\begin{array}{c} (\;.\;.\;.\;.\;.\;.\;) \\ \perp \end{array}$$

Remark: $\lfloor . \rfloor$ is continuous (if D is a cpo)

Strict extension of functions from D to $D_\perp$: (**Definition:** f is strict iff. $f(\perp) = \perp$)

$f : [D \to E], D$ cpo, E cpo with $\perp$
$f^* : [D_\perp \to E]$
$f^*(d') = \begin{cases} \perp & \text{if } d' = \perp \\ f(d) & \text{if } d' = \lfloor d \rfloor \end{cases}$
f^* is the strict extension of f.

Notation: x variable, d, e are terms. $e \in E$, $d \in D_\perp$

let $x \Leftarrow d.e \equiv (\lambda x.e)^* d$
let $x_1 \Leftarrow d_1, \ldots, x_n \Leftarrow d_n.e \equiv$ **let** $x_1 \Leftarrow d_1 \ldots$ **let** $x_n \Leftarrow d_n.e$

Example:

$$\begin{array}{rcl} \textbf{let } x \Rightarrow \lfloor 5 \rfloor . x+2 & = & (\lambda x.x+2)^* \lfloor 5 \rfloor \\ & = & (\lambda x.x+2) 5 = 7 \\ \perp & = & \perp_{D_\perp} = \perp_E \end{array}$$

Given $\otimes : S \times S \to S$
define $\otimes_\perp : S_\perp \times S_\perp \to S_\perp$
$s_1' \otimes s_2' =$ **let** $x_1 \Rightarrow s_1', x_2 \Rightarrow s_2'. x_1 \otimes x_2$

Miscellaneous functions

$cond : T \times D \times D \to D, D$ cpo
$cond(\text{tt}, x, y) = x$
$cond(\text{ff}, x, y) = y$
is continuous.

Proof:
First argument: T discrete
Second argument: $\lambda x.cond(b, x, y)$ is either $\lambda x.x$ (if $b = \text{tt}$) or $\lambda x.y$ – both are continuous.

update: $\Sigma \times Var \times \mathbb{Z} \to \Sigma$
$\sigma[x \mapsto n]$ is continuous because $\Sigma \times Var \times \mathbb{Z}$ is discrete.

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fix: $\big[[D \to D] \to D\big]$ (continuity of fix needs proof!) [WINSKEL93]

Theorem: (Winskel)
Well-typed λ-terms containing only continuous functions are continuous (in their free variables).

⇒ No more worry about continuity, as long as basic functions are continuous.

⇒ Every recursion equation $x = \Gamma(x)$ has a least solution $x = fix(\Gamma)$

4.4 A cpo-based denotational semantics for WHILE

$\Sigma \to \Sigma_\perp \cong \Sigma \nrightarrow \Sigma$

$$\begin{array}{rcl}
D : Stm & \to & (\Sigma \to \Sigma_\perp) \\
D[\,\textbf{skip}\,] & = & \lambda\sigma.\lfloor\sigma\rfloor \\
D[x := a] & = & \lambda\sigma.\lfloor\sigma[x \mapsto A[a]\sigma]\rfloor \quad A \text{ continuous because } Aexp \text{ and } \Sigma \text{ are discrete} \\
D[s_1; s_2] & = & \big(\lambda\sigma.\textbf{let } \sigma' = D[s_1]\sigma^{15}.D[s_2]\sigma'\big) = D[s_2] \circ_\perp D[s_1]^{9} \\
D[\,\textbf{if } b \textbf{ then } s_1 \textbf{ else } s_2] & = & \lambda\sigma.cond(B[b]\sigma, D[s_1]\sigma, D[s_2]\sigma\,) \\
D[\,\textbf{while } b \textbf{ do } s\,] & = & \text{least solution of} \\
 & & \Phi = \lambda\sigma.cond\big(B[b]\sigma, (\Phi \circ_\perp D[s])\sigma, \lfloor\sigma\rfloor\big) \\
 & = & fix\underbrace{\big(\lambda\Phi.\lambda\sigma.cond\,(B[b]\sigma, (\Phi \circ_\perp D[s])\sigma, \lfloor\sigma\rfloor)\big)}_{\Gamma}
\end{array}$$

Example:

$s = \textbf{while } i \neq 0 \textbf{ do } i := i - 1$

$$\begin{array}{rcl}
D[s]\{i \mapsto 2\} & = & fix(\Gamma)\{i \mapsto 2\} \\
 & = & \Gamma(fix(\Gamma))\{i \mapsto 2\} \quad \text{fix unfolded once} \\
 & = & \Big(\lambda\sigma.cond\big(\,B[i \neq 0]\{i \mapsto 2\}, fix(\Gamma) \circ_\perp D[i := i-1]\,\big)\{i \mapsto 2\}, \quad \underbrace{\lfloor\{i \mapsto 2\}\rfloor}_{\text{argument for } \lambda\sigma....}\Big) \\
 & = & \textbf{let } \sigma' = D[i := i-1]\{i \mapsto 2\}.fix(\Gamma)\sigma' \\
 & = & fix(\Gamma)\{i \mapsto 1\} \\
 & & \vdots \\
 & = & fix(\Gamma)\{i \mapsto 0\} \\
 & = & cond(B[i \neq 0]\{i \mapsto 0\}, \ldots, \lfloor\{i \mapsto 0\}\rfloor) \\
 & = & \lfloor\{i \mapsto 0\}\rfloor
\end{array}$$

[15] $D[s_1]\sigma \in \Sigma_\perp$

[9] $F : C \to D_\perp$
$g : D \to E$
$g \circ_\perp f = \lambda c.\textbf{let } x = f\,c.g\,x$
where $x \in D$ and $f \in D_\perp$

4.5 Extensions of WHILE

4.5.1 Local vars and procedures

Only static scoping!

$$\begin{array}{rcl} \text{Syntax} & : & \{\mathbf{var}\ x := a;\ s\} \\ & & \{\mathbf{proc}\ p = s_1;\ s_2\} \\ & & \mathbf{call}\ p \end{array}$$

Semantic domains:

$$\begin{array}{rcl} Venv & = & Var \overset{fin}{\rightarrow} Loc\ (\text{ve}) \\ Store & = & Loc \overset{fin}{\rightarrow} \mathbb{Z}\ (\sigma) \\ & & \sigma \circ ve \in \Sigma = Var \rightarrow \mathbb{Z} \end{array}$$

Procedure environment records semantics, not syntax!

$$Penv = Pname \overset{fin}{\rightarrow} (Store \rightarrow Store_\perp)$$

$$\begin{array}{rcl} D : Stm \rightarrow Venv \rightarrow Penv & \rightarrow & (Store \rightarrow Store_\perp) \\ D[\ \mathbf{skip}\ ]\ ve\ pe & = & \lambda\sigma.\lfloor\sigma\rfloor \\ D[x := a]\ ve\ pe & = & \lambda\sigma.\lfloor\sigma[ve(x) \mapsto A[a](\sigma \circ ve)]\rfloor \\ D[s_1;\ s_2]\ ve\ pe & = & D[s_2]\ ve\ pe \circ_\perp D[s_1]\ ve\ pe \\ D[\mathbf{if}\ b\ \mathbf{then}\ s_1\ \mathbf{else}\ s_2]\ ve\ pe & = & \lambda\sigma.cond(B[b](\sigma \circ ve), D[s_1]\ ve\ pe,\ \ .\overset{s_2}{.}.) \\ D[w]\ ve\ pe & = & \lambda\sigma.cond(B[b](\sigma \circ ve), (D[w]\ ve\ pe \circ_\perp D[s]\ ve\ pe)\sigma, \lfloor\sigma\rfloor) \\ D[\mathbf{call}\ p]\ ve\ pe & = & pe(p)\quad \text{(context condition: } pe(p) \text{ is defined,} \\ & & \qquad \text{i.e. all called procedures are defined in} \\ & & \qquad \text{the program text)} \\ D[\{\mathbf{var}\ x = a;\ s\}]\ ve\ pe & = & \lambda\sigma.D[s](ve[x \mapsto l])\ pe\ (\sigma[l \mapsto A[a](\sigma \circ ve)]) \\ & & \text{where } l \text{ is new, i.e. } l \notin dom(\sigma) \\ D[\{\mathbf{proc}\ p = s_1; s_2\}]\ ve\ pe & = & D[s_2]\ ve\ (pe[p \mapsto \Phi]) \\ & & \text{where } \Phi = D[s_1]\ ve\ (pe[p \mapsto \Phi])\quad \text{(least solution!)} \end{array}$$

Example:

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$$s = \{\mathbf{proc}\ p = \underbrace{(\mathbf{if}\ i = 0\ \mathbf{then}\ \mathbf{skip}\ \mathbf{else}\ (i := i - 1;\ \mathbf{call}\ p))}_{s_p};$$
$$\mathbf{call}\ p\}$$

$$\begin{array}{rcl} D[s]\{i \mapsto l\}\ \{\}\ \{l \mapsto 1\} & = & D[\mathbf{call}\ p]\{i \mapsto l\}\ \{p \mapsto \Phi\}\ \{l \mapsto 1\} \text{ where } \Phi = D[s_p]\{i \mapsto l\}\ \{p \mapsto \Phi\} \\ & = & \Phi\{l \mapsto 1\} = D[s_p]\{i \mapsto l\}\ \{p \mapsto \Phi\}\ \{l \mapsto 1\} \\ & = & D[i := i - 1;\ \mathbf{call}\ p]\{i \mapsto l\}\ \{p \mapsto \Phi\}\ \{l \mapsto 1\} \\ & = & \mathbf{let}\ \sigma' = D[i := i - 1]\{i \mapsto l\}\ \{p \mapsto \Phi\}\ \{l \mapsto 1\}.D[\mathbf{call}\ p]\{i \mapsto l\}\ \{p \mapsto \Phi\}\ \sigma' \\ & = & D[\mathbf{call}\ p]\{i \mapsto l\}\ \{p \mapsto \Phi\}\ \{l \mapsto 0\} \\ & = & \Phi\{l \mapsto 0\} = D[s_p]\{i \mapsto l\}\ \{p \mapsto \Phi\}\ \{l \mapsto 0\} \\ & = & \lfloor\{l \mapsto 0\}\rfloor \in Store_\perp \end{array}$$

4.5.2 Continuations and Exceptions

Continuations = (meaning of the) continuing computation = functional gotos
New statements for **WHILE** :

$$\textbf{escape}$$
$$s_1 \textbf{ handle } s_2$$

Example: (**if** $x = 0$ **then escape else** $x := x/x$) **handle** $x := 1$

Problem: $D[\textbf{escape}\,] = ?? \in \Sigma \to \Sigma_\perp$
Solution: continuations for normal and exceptional behavior.

Definition:
$Cont = \Sigma \to \Sigma_\perp$ = meaning of continuous computation
$D : Stm \to \underbrace{Cont}_{normal} \to \underbrace{Cont}_{exceptional} \to Cont$

Informal: $D[s]\ c\ d$ = execute s and continue $\begin{cases} \text{normally with c} \\ \text{exceptionally with d} \end{cases}$

$$\begin{aligned}
D[\,\textbf{skip}\,]\ c\ d &= c = \lambda\sigma.c\ \sigma \\
D[\,\textbf{escape}\,]\ c\ d &= d \\
D[x := a]\ c\ d &= \lambda\sigma.c(\sigma[x := A[a]\sigma]) \\
D[s_1;\ s_2]\ c\ d &= D[s_1](D[s_2]\ c\ d)\ d \\
D[s_1 \textbf{ handle } s_2]\ c\ d &= D[s_1]\ c\ (D[s_2]\ c\ d) \\
D[\textbf{if } b \textbf{ then } s_1 \textbf{ else } s_2]\ c\ d &= \lambda\sigma.cond(B[b]\sigma, D[s_1]\ c\ d\ \sigma, \ldots s_2 \ldots) \\
D[w]\ c\ d &= \lambda\sigma.cond(B[b]\sigma, D[s](D[w]\ c\ d)\ d\ \sigma, c\ \sigma)
\end{aligned}$$

Example:

$$\begin{aligned}
& D[(x := 1; \textbf{escape}\ ; x := 2) \textbf{ handle } x := 3]\ c\ d\ \sigma \\
= \ & D[(x := 1; \textbf{escape}\ ; x := 2) \textbf{ handle } x := 3]\ c\ \underbrace{(D[x := 3]\ c\ d)}_{d'}\ \sigma \\
= \ & D[x := 1](D[\textbf{escape}\ ; \ldots]\ c\ d')\ d'\ \sigma \\
= \ & D[\textbf{escape}\ ; \ldots]\ c\ d')\ d'\ (\sigma[x \mapsto 1]) \\
= \ & D[\textbf{escape}\,]\ (\ldots)\ d'\ (\sigma[x \mapsto 1]) = d'\ (\sigma[x \mapsto 1]) \\
= \ & D[x := 3]\ c\ d\ (\sigma[x \mapsto 1]) \\
= \ & c\ (\sigma[x \mapsto 3])
\end{aligned}$$

Top level call of D :
$D[s]\ (\lambda\sigma.\lfloor\sigma\rfloor)\ (\lambda\sigma.\perp)$ or
$D[s]\ (\lambda\sigma.\lfloor\sigma\rfloor)\ (\lambda\sigma.\lfloor\sigma[\underbrace{excn}_{Var} \mapsto 1]\rfloor)$

4.5.3 The Knaster-Tarski Fixpoint theorem

cpo $\supseteq$ complete lattice
continuity $\subseteq$ monotony

Definition:
Let $(P, \sqsubseteq)$ be a partial order, $X \subseteq P$.

- $p \in P$ is a *lower bound* for X iff. $\forall x \in X . p \sqsubseteq x$.
- $p \in P$ is a *greatest lower bound* for X iff. p is a lower bound and $q \sqsubseteq p$ for all lower bounds q of X.

(infimum = greatest lower bound)

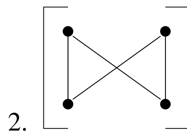
$\bigsqcap X$ = infimum of X, if it exists.

Definition: A partial order is a *complete lattice*[16] iff. every subset of P has an infimum.

Example:
$\wp A$ is a complete lattice with $\bigsqcap = \bigcap$.

$\bigcap \emptyset = A$ because any set $B \subseteq A$ is a lower bound for $\emptyset$, and thus A is the greatest lower bound.

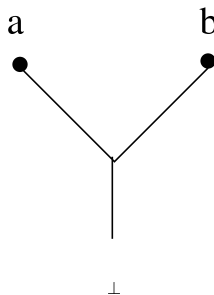
Counter-Example

1. [• •]
2. 
   Be careful! $P(A) : \bigsqcap \emptyset = A$!!! Because any set $B \subseteq A$ is a lower bound for $\emptyset$.

Remarks: If all infima exist, then all suprema exist as well:

$$\bigsqcup X = \bigsqcap \{u \in P \mid u \text{ is upper bound for } X\}$$

$$\begin{aligned}\bigcup \big\{\{1,2\},\{3,4\}\big\} &= \bigcap \big\{U \in \wp(\{1,\ldots,5\}) \mid U \supseteq \{1,2\},\{3,4\}\big\} \\ &= \bigcap \big\{\{1,2,3,4\},\{1,2,3,4,5\}\big\} = \{1,2,3,4\}\end{aligned}$$



$\bigsqcap \emptyset$ is the greatest element

[16] "Vollständiger Verband"

$\bigsqcup\emptyset$ is the least element

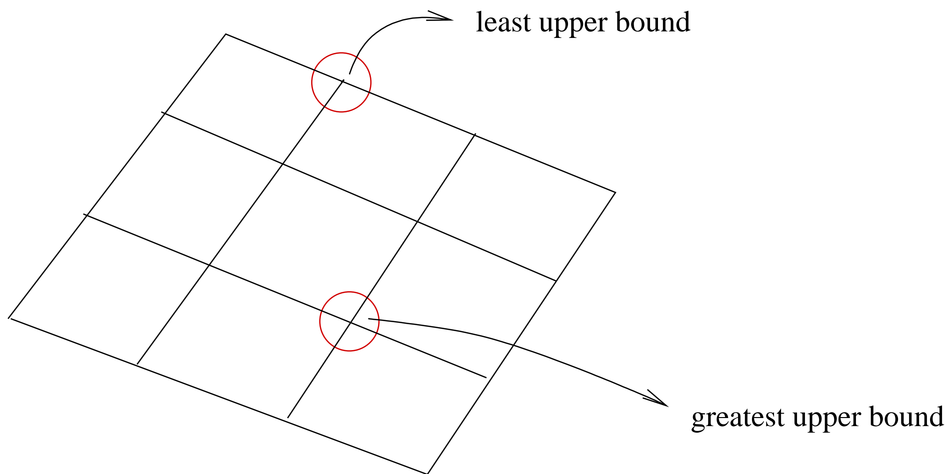
Trick Question:

- is $(\mathbb{N},\leq)$ a complete lattice???
- infimum: take least element in the subset
- supremum of $\mathbb{N}$???

Answer:

- $\bigsqcap\emptyset$ does not exist
- $\bigsqcup\mathbb{N}$ does not exist

$\Rightarrow$ not a complete lattice



Theorem:
Let $(L,\sqsubseteq)$ be a complete lattice, $f : L \to L$ be monotone.
Then f has a least fixpoint $m := \bigsqcap\{x \in L \mid f(x) \sqsubseteq x\}$ $\bigsqcap$ greatest lower bound

(Then $\{x \in L \mid f(x) = x\}$ is again a complete lattice.)

Proof: $f(m) = m, X := \{x \in L \mid f(x) \sqsubseteq x\}$

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1. $f(m) \sqsubseteq m$

   Let $x \in X \Rightarrow m \sqsubseteq x \overset{f \text{ monotone}}{\Rightarrow} f(m) \sqsubseteq f(x) \sqsubseteq x$
   $\Rightarrow f(m)$ is lower bound of X
   $\overset{m \text{ greatest lower bound}}{\Rightarrow} f(m) \sqsubseteq m$

2. $m \sqsubseteq f(m)$

   $f(m) \sqsubseteq m \overset{f \text{ monotone}}{\Rightarrow} f(f(m)) \sqsubseteq f(m)$
   $\Rightarrow f(m) \in X$
   $\overset{m=\bigsqcap X}{\Rightarrow} m \sqsubseteq f(m)$

3. m is *least* fixpoint:

   Let $f(p) = p$ be another fixpoint of f. Then $m \sqsubseteq p$:
   $f(p) = p \Rightarrow f(p) \sqsubseteq p \overset{\text{property of } X}{\Rightarrow} p \in X$
   $\overset{glb}{\Rightarrow} m \sqsubseteq p$

4.5.4 Nondeterminism

Extension of **WHILE** with nondeterministic choice:

$$s_1 \textbf{ or } s_2$$

Semantics: $S[s_1 \textbf{ or } s_2] = S[s_1] \cup S[s_2]$
where $S : Stm \rightarrow 2^{\Sigma \times \Sigma}$ is the relational semantics from section 4. There $S : Stm \rightarrow (\Sigma \nrightarrow \Sigma)$, but $\Sigma \nrightarrow \Sigma \subseteq 2^{\Sigma \times \Sigma}$ (Alternative: $Stm \rightarrow \Sigma \rightarrow 2^{\Sigma}$)
So what's new?
Example: $w_0 = \textbf{while } x \neq 0 \textbf{ do } (x := 0 \textbf{ or skip })$

$$\begin{aligned} S[w_0] &= \{(\sigma, \sigma[x \mapsto 0]) \mid \sigma \in \Sigma\} \\ S[x := 0] &= \{(\sigma, \sigma[x \mapsto 0]) \mid \sigma \in \Sigma\} \end{aligned}$$

w_0 may not terminate if **or** is "unfair".

Problem: S does not distinguish between *possible* termination (w_0) and *guaranteed* termination ($x := 0$)
Solution: $T : Stm \rightarrow 2^{\Sigma}$
$T[s]$ = set of start states that guarantee termination of s

$$\begin{aligned} T[\textbf{skip}] &= \Sigma \\ T[x := a] &= \Sigma \\ T[s_1 \textbf{ or } s_2] &= T[s_1] \cap T[s_2] \\ T[s_1; s_2] &= \{\sigma \in T[s_1] \mid \forall \sigma' (\sigma, \sigma') \in S[s_1] \Rightarrow \sigma' \in T[s_2]\} \\ &= \{\sigma \in T[s_1] \mid S[s_1]\sigma \subseteq T[s_2]\}^{17} \\ T[\textbf{if } b \textbf{ then } s_1 \textbf{ else } s_2] &= \{\sigma \mid \mathcal{B}[b]\sigma = \text{tt} \Rightarrow \sigma \in T[s_1] \wedge \mathcal{B}[b]\sigma = \text{ff} \Rightarrow \sigma \in T[s_2]\} \\ T[\underbrace{\textbf{while } b \textbf{ do } s}_{w}] &= T[\textbf{if } b \textbf{ then } s; w \textbf{ else skip }] \\ &= \{\sigma \mid \mathcal{B}[b]\sigma = \text{tt} \Rightarrow (\sigma \in T[s] \wedge S[s]\sigma \subseteq T[w]\} \quad \text{least solution} \\ &= fix(\Delta) \end{aligned}$$

where $\Delta : 2^{\Sigma} \rightarrow 2^{\Sigma}$
$\Delta(\mu) = \{\sigma \mid \mathcal{B}[b]\sigma = \text{tt} \Rightarrow (\sigma \in T[s] \wedge S[s]\sigma \subseteq \mu\}$
Does Δ have a least fixpoint? Yes!
Apply the KT-FP-theorem because $(2^{\Sigma}, \subseteq)$ is a complete lattice and Δ is monotone:
$\mu_1 \subseteq \mu_2 \Rightarrow \Delta(\mu_1) \subseteq \Delta(\mu_2)$

In some cases Δ is even continuous.

---

[17] Convention: If $R \subseteq A \times B, a \in A, X \subseteq A$: $R(a) = \{b \mid (a,b) \in R\}, R(X) = \{b \mid \exists a \in X.(a,b) \in R\} = \bigcup_{\text{ßn}X} R(a)$

Lemma: If $S[s]\sigma$ is finite ("finite nondeterminism") for all σ, then Δ is continuous.

Proof: Exercises!

In fact: Δ always continuous, but

- difficult to prove
- not true for other nondeterministic constructs, e.g. $random(x)$

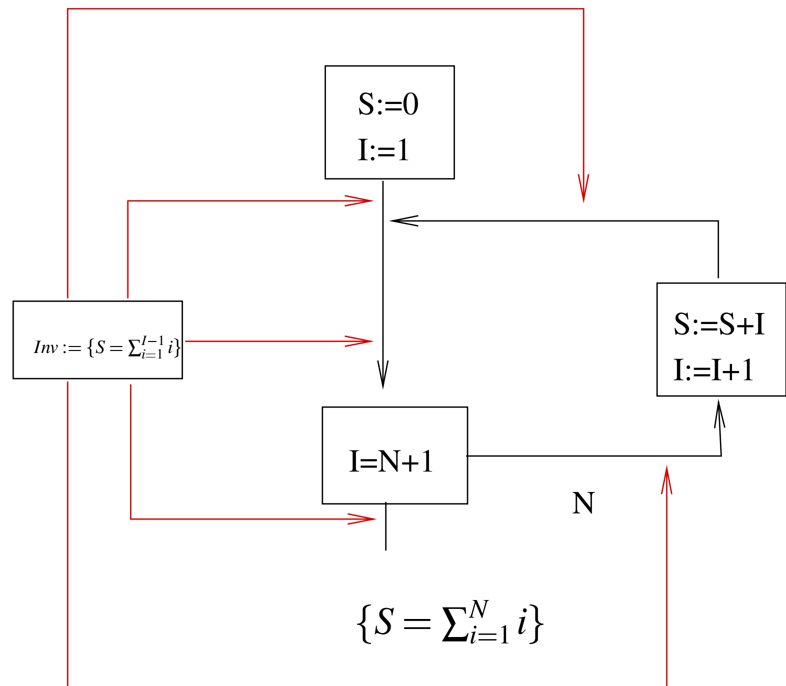
Example: $w_0 = \textbf{while } x \neq 0 \textbf{ do } (x := 0 \textbf{ or skip })$
$T[w_0] = ?$
Compute $fix(\Delta)$ by iteration: $\emptyset, \Delta(\emptyset), \Delta^2(\emptyset), \ldots$
Maybe find fixpoint in finite time / number of steps

$$\begin{aligned}
\Delta(\mu) &= \{\sigma \mid \sigma(x) \neq 0 \Rightarrow \sigma \in T[\ldots \textbf{ or } \ldots] \wedge S[\ldots \textbf{ or } \ldots]\sigma \subseteq \mu\} \\
&= \{\sigma \mid \sigma(x) \neq 0 \Rightarrow \{\sigma, \sigma[x \mapsto 0]\} \subseteq \mu\} \\
\Delta(\emptyset) &= \{\sigma \mid \sigma(x) = 0\} \\
\Delta^2(\emptyset) &= \Delta(\Delta(\emptyset)) \\
&= \Big\{\sigma \mid \sigma(x) \neq 0 \Rightarrow \underbrace{\{\sigma, \sigma[x \mapsto 0]\} \subseteq \{\sigma \mid \sigma(x) = 0\}}_{false}\Big\} \\
&= \{\sigma \mid \sigma(x) = 0\} = \Delta(\emptyset) = fix(\Delta)
\end{aligned}$$

Denotational Semantics for parallelism: tricky!

5 Axiomatic Semantics (Hoare-Logic; 1969)

01/09/2002



Invariant: $Inv = \{S = \sum_{i=1}^{I-1} i\}$

If we show that Inv holds at the last point (N), then the result is correct. To prove: If Inv is true before B, then Inv is true after B.

Assume Inv. To show: $S' = \sum_{i=1}^{I'-1} i$ where S', I' are the values of S, I after B.

$$\begin{aligned} S' &= S + I \\ &\overset{Inv}{=} \sum_{i=1}^{I-1} i \\ &= \sum_{i=1}^{I} i \\ &= \sum_{i=1}^{I'-1} i \text{ qed} \end{aligned}$$

Hoare logic:

$$S := 0; I := 1$$
$$\{S = \sum_{i=1}^{I-1} i\}$$
$$\textbf{while } I \neq N+1 \textbf{ do } (S := S+I; I := I+1)$$
$$\{S = \sum_{i=1}^{N} i\}$$

Hoare triple

$$\{\text{precondition}\} \text{ statement } \{\text{postcondition}\}$$

Intuitive meaning of $\{P\}S\{Q\}$:
if P is true in state σ, and execution of s takes σ to σ', then Q is true in σ'.

Partial correctness of $\{P\}S\{Q\}$

$$\{x = 0\}\textbf{while true do } x := x+1\{x = 0\}$$

is partially correct. Total correctness = partial correctness and termination

Examples of correct Hoare triples:

$$\begin{array}{ccl} \{I = 5\} & I := I+5 & \{I = 10\} \\ \{\textbf{ true }\} & I := 10 & \{I = 10\} \\ \{x = y\} & x := x+1 & \{x \neq y\} \text{ if } x \text{ and } y \text{ are distinct variables} \end{array}$$

Boundary cases

$\{$ **true** $\}$ s $\{$ **true** $\}$
partially correct for every s, total correct for any terminating s

$\{$ **true** $\}$ s $\{$ **false** $\}$
partially correct for nonterminating s, total correct for no s

$\{$ **false** $\}$ s $\{P\}$
partially correct for every s, total correct for any terminating s

The rules of Hoare logic (partial correctness):

$$\{P\} \quad \textbf{skip} \quad \{P\}$$
$$\underbrace{\{P[a/x]\}}_{P \text{ with } x \text{ replaced by } a} \quad x := a \quad \{P\}$$

$$\{x+2 = 7\}x := x+2\{x = 7\}$$

$$\frac{\{P\}s_1\{Q\} \qquad \{Q\}s_2\{R\}}{\{P\}s_1; s_2\{R\}}$$

$$\frac{\{P \wedge b\}s_1\{Q\} \qquad \{P \wedge \neg b\}s_2\{Q\}}{\{P\}\textbf{if } b \textbf{ then } s_1 \textbf{ else } s_2 \{Q\}}$$

$$\frac{\{P \wedge b\} s \{P\} \qquad \text{"}P \text{ is invariant"}}{\{P\} w \{P \wedge \neg b\}}$$

$$\frac{P \Rightarrow P' \qquad \{P'\} s \{Q'\} \qquad Q' \Rightarrow Q}{\{P\} s \{Q\}}$$

strengthening precondition, weakening postcondition

$$\frac{\{x = 0\} s \{x = 5\}}{\{\ \textbf{false}\ \} s \{x < 10\}}$$

Derived rules

$$\frac{P \Rightarrow Q[a/x]}{\{P\} x := a \{Q\}}$$

$$\frac{x = 5 \Rightarrow (x+2) = 7}{\{x = 5\} x := x+2 \{x = 7\}}$$

$$\frac{P \Rightarrow Q[a/x] \qquad \{Q[a/x]\} x := a \{Q\} \qquad Q \Rightarrow Q}{\{P\} x := a \{Q\}}$$

$$\frac{P \Rightarrow Inv \qquad \{Inv \wedge b\} s \{Inv\} \qquad Inv \wedge \neg b \Rightarrow Q}{\{P\} w \{Q\}}$$

$$
\text{; rule}\ \dfrac{
\dfrac{\checkmark}{\{\ \mathbf{true}\ \}S := 0; I := 1\{S = \sum_{i=1}^{I-1} i\}}
\qquad
\dfrac{
S = \sum_{i=1}^{I-1} i \Rightarrow S = \sum_{i=1}^{I-1} i
\qquad
\dfrac{
\dfrac{S = \sum_{i=1}^{I-1} i \wedge I \neq N+1 \Rightarrow S+I = \sum_{i=1}^{I} i}{\{S = \sum_{i=1}^{I-1} i \wedge I \neq N+1\}S := S+I\{S = \sum_{i=1}^{I} i\}}
\qquad
\{S = \sum_{i=1}^{I} i\}I := I+1\{Inv\}
}{\{S = \sum_{i=1}^{I-1} i \wedge I \neq N+1\}S := S+I; I := I+1\{S = \sum_{i=1}^{I-1} i\}}
\qquad
S = \sum_{i=1}^{I-1} i \wedge \neg(I \neq N+1) \Rightarrow S = \sum_{i=1}^{N} i
}{\{S = \sum_{i=1}^{I-1} i\}\mathbf{while}\ \ldots\ \{S = \sum_{i=1}^{N} i\}}
}{\{\ \mathbf{true}\ \}S := 0; I := 1; \mathbf{while}\ I \neq N+1\ \mathbf{do}\ (S := S+I; I := I+1)\{S = \sum_{i=1}^{N} i\}}
$$

5.1 Assertions

"Zusicherungen"

The *syntactic* (intensional) approach [WINSKEL93]:

- assertions = syntactic formulae from some logical system (usually: FOL[18] + $\mathbb{Z}$ = first order arithmetic)

The *semantic* (extensional) approach [NIELSON99]:

- assertions = predicates on $\Sigma = \Sigma \to T \quad (T = \{\text{tt}, \text{ff}\})$

Notation: abbreviate $(P = \text{tt})$ by P.

Example:

$$\begin{aligned} \text{syntactic} &: \; x > y \\ \text{semantic} &: \; \lambda\sigma.\sigma(x) > \sigma(y) : \Sigma \to T \end{aligned}$$

Note: $\Sigma \to T$ is the set of *all* functions, not just those expressible in some fixed language (e.g. *Bexp*)

5.2 The semantic approach

Definition: (Validity/Truth)

$$\begin{aligned} \models \{P\}s\{Q\} \quad &\Leftrightarrow \quad \forall \sigma, \sigma'. P(\sigma) \wedge \langle s, \sigma \rangle \to \sigma' \Rightarrow Q(\sigma') \\ &\Leftrightarrow \quad \forall \sigma, \sigma'. \langle s, \sigma \rangle \to \sigma' \Rightarrow (P\,\sigma \Rightarrow Q\sigma') \end{aligned}$$

"$\{P\}s\{Q\}$ is valid"

$\vdash \{P\}s\{Q\}$ means the triple $\{P\}s\{Q\}$ is *provable/derivable* in the following system:

$$\vdash \{P\}\textbf{skip}\ \{P\}$$

;-rule unchanged

$$\vdash \frac{\{\lambda\sigma.P\sigma \wedge \mathcal{B}[b]\sigma\}s_1\{Q\}}{\{P\}\textbf{if}\ b\ \textbf{then}\ s_1\ \textbf{else}\ s_2\{Q\}}$$

$$\vdash \frac{\{\lambda\sigma.P\ \sigma \wedge \mathcal{B}[b]\sigma\}s\{P\}}{\{P\}w\{\lambda\sigma.P\ \sigma \wedge \neg\mathcal{B}[b]\sigma\}}$$

before: $\vdash \{P[a/x]\}x := a\{P\}$

now: $\vdash \{\underbrace{\lambda\sigma.P(\sigma[x \mapsto \mathcal{A}[a]\sigma])}_{P[x \mapsto a]}\}x := a\{P\}$

$$\frac{\forall\sigma.P\sigma \Rightarrow P'\sigma \qquad \vdash \{P'\}s\{Q'\} \qquad \forall\sigma.Q'\sigma \Rightarrow Q\sigma}{\vdash \{P\}s\{Q\}}$$

Wanted: soundness: $\vdash \{P\}s\{Q\} \Rightarrow \;\models \{P\}s\{Q\}$
completeness: $\vdash \{P\}s\{Q\} \Leftarrow \;\models \{P\}s\{Q\}$

01/14/2002

5.3 Soundness

Theorem: $\vdash \{P\}s\{Q\} \Rightarrow \ \models \{P\}s\{Q\}$

Proof: by rule induction on $\vdash \{P\}s\{Q\}$,
i.e. by showing that each rule preserves validity ($\models$)
Two cases:

- **:=** To show $\models \{P[x \mapsto a]\}x := a\{P\}$

$$\begin{aligned} &\Leftrightarrow \quad \forall \sigma, \sigma'. \langle x := a, \sigma\rangle \to \sigma' \Rightarrow (P(\sigma[x \mapsto \mathcal{A}[a]\sigma]) \Rightarrow P\sigma') \\ &\Leftrightarrow \quad \forall \sigma, \sigma'. \sigma' = \sigma[x \mapsto \mathcal{A}[a]\sigma] \Rightarrow (P(\sigma[x \mapsto \mathcal{A}[a]\sigma]) \Rightarrow P\sigma') \quad \checkmark \end{aligned}$$

- **while** :

  (A) Assume $\models \{\lambda\sigma.P\sigma \wedge \mathcal{B}[b]\sigma\}s\{P\}$

  (B) To show: $\models \{P\}w\{\lambda\sigma.P\sigma \wedge \neg\mathcal{B}[b]\sigma\}$

  Lemma: If $\langle w, \sigma\rangle \to \sigma', P\sigma, \forall \sigma, \sigma'. P\sigma \wedge \mathcal{B}[b]\sigma \wedge \langle s, \sigma\rangle \to \sigma' \Rightarrow P\sigma'$ then $P\sigma' \wedge \neg\mathcal{B}[b]\sigma'$

  Proof: by induction on $\langle w, \sigma\rangle \to \sigma'$
  Proof of (B): Assume $\langle w, \sigma\rangle \to \sigma', P\sigma$. Then by lemma
  and (A) we have $P\sigma' \wedge \neg\mathcal{B}[b]\sigma'$ qed.

5.4 Completeness

Theorem: $\models \{P\}s\{Q\} \Rightarrow \vdash \{P\}s\{Q\}$
Proof idea based on weakest preconditions.

Definition:

$$\begin{aligned} wp\ s\ Q &= \lambda\sigma.\forall\sigma'.\langle s, \sigma\rangle \to \sigma' \Rightarrow Q\ \sigma' \\ &= \text{the set of all } \sigma \text{ that lead into } Q \text{ when executing } s \end{aligned}$$

(often: weakest *liberal* precondition)

Fact:

- $\models \{P\}s\{Q\}$ iff. $\forall\sigma P\sigma \Rightarrow wp\ s\ Q\sigma$

Lemmas:

- wp **skip** $Q = Q$
- $wp\ (x := a)\ Q = Q[x \mapsto a]$
- $wp\ (s_1; s_2)\ Q = wp\ s_1\ (wp\ s_2\ Q)$

- $wp\ (\textbf{if}\ b\ \textbf{then}\ s_1\ \textbf{else}\ s_2)Q = (\lambda\sigma.(\mathcal{B}[b]\sigma \Rightarrow wp\ s_1\ Q\sigma) \wedge (\neg\mathcal{B}[b]\sigma \Rightarrow wp\ s_2\ Q\sigma))$
- $wp\ (w)\ Q\sigma = \begin{cases} Q\sigma & \text{if } \neg\mathcal{B}[b]\sigma \\ wp\ (s;w)\ Q\sigma = wp\ s\ (wp\ w\ Q)\sigma & \text{if } \mathcal{B}[b]\sigma \end{cases}$

$$\text{consequence rule}\ \frac{\dfrac{\models \{P\}s\{Q\}}{\forall\sigma.P\sigma \Rightarrow wp\ s\ Q\sigma} \qquad \dfrac{\text{Lemma}}{\vdash \{wp\ s\ Q\}s\{Q\}} \qquad \overset{\checkmark}{\forall\sigma.\ Q\sigma \Rightarrow Q\sigma}}{\vdash \{P\}s\{Q\}}$$

Lemma: $\forall Q. \vdash \{wp\ s\ Q\}\ s\ \{Q\}$

Proof: by induction on s

1. $\vdash \{wp\ \textbf{skip}\ Q\}\ \textbf{skip}\ \{Q\}$ ✓
2. $\vdash \{wp\ (x := a)\ Q\}\ x := a\ \{Q\}$ ✓
3. $$\frac{\overset{IH}{\vdash \{wp\ s_1\ (wp\ s_2\ Q)\}\ s_1\ \{wp\ s_2\ Q\}} \qquad \overset{IH}{\vdash \{wp\ s_2\ Q\}\ s_2\ \{Q\}}}{\{wp\ (s_1;s_2)\ Q\}\ s_1;s_2\ \{Q\}}$$
4. $$\frac{\vdash \{\lambda\sigma.\overbrace{\mathcal{B}[b]\sigma \wedge wp\ (\textbf{if}\ \dots)\ \sigma}^{\mathcal{B}[b]\sigma\wedge wp\ s_1\ Q\sigma}\}\ s_1\ \{Q\}}{\{wp\ (\textbf{if}\ \dots)\ Q\}\ \textbf{if}\ \dots\ \{Q\}}$$

$$IH_1:\ \text{conseq.}\ \frac{\vdash \{wp\ s_1\ Q\}\ s_1\ \{Q\}}{\vdash \{\lambda\sigma.\mathcal{B}[b]\sigma \wedge wp\ s_1\ Q\sigma\}\ s_1\ \{Q\}}$$

5. $$\frac{\dfrac{\text{conseq.}\ \dfrac{\overset{IH}{\vdash \{wp\ s\ (wp\ w\ Q)\}\ s\ \{wp\ w\ Q\}}}{\vdash \{wp\ w\ Q \wedge \mathcal{B}[b]\sigma\}\ s\ \{wp\ w\ Q\}}}{\vdash \{wp\ w\ Q\}\ w\ \{\lambda\sigma.wp\ w\ Q\sigma \wedge \neg\mathcal{B}[b]\sigma\}}}{\vdash \{wp\ w\ Q\}\ w\ \{Q\}}$$

$$\begin{aligned} & wp\ w\ Q\sigma \wedge \neg\mathcal{B}[b]\sigma \\ \overset{Lemma}{=}\ & Q\sigma \wedge \neg\mathcal{B}[b]\sigma \Rightarrow Q\sigma \end{aligned}$$

$$\begin{aligned} & wp\ w\ Q\sigma \wedge \mathcal{B}[b]\sigma \\ \overset{Lemma}{=}\ & wp\ s\ (wp\ w\ Q)\sigma \wedge \mathcal{B}[b]\sigma \\ \Rightarrow\ & wp\ s\ (wp\ w\ Q)\sigma \end{aligned}$$

5.5 Expressiveness

Problem: Proof of $\vdash \{P\}s\{Q\}$ with $P,Q \in A$ (A set of formulae) may require assertions not in A.

Contrived example: $A = \{\textbf{true}\} \cup \{X = n \mid X \in \textbf{var} \wedge n \in \mathbb{Z}\}$
(*Bexp* are restricted to A)

$$
\begin{array}{rcl}
\models \{\textbf{true}\} & & \\
 & & \textbf{if } y = 0 \textbf{ then } x := 0 \textbf{ else } x := 1 \\
\{x = 0 \vee x = 1\} & \leftarrow & \text{not in } A \\
 & & \textbf{if } x = 0 \textbf{ then skip} \\
 & & \textbf{else if } x = 1 \textbf{ then } x := 0 \textbf{ else } x := 2 \\
\{x = 0\} & & \\
 & & \Rightarrow \text{ not provable!}
\end{array}
$$

Another example: Presburger arithmetic = FOL + $(\mathbb{N}, +)$
Problems: programs can synthesize $*$ from $+$, but not FOL

$$
\begin{array}{rcl}
\{p = 0 \wedge x \geq 0 \wedge y \geq 0\} & & \\
 & & \textbf{while } x > 0 \textbf{ do } (p := p + y;\ x := x - 1) \\
\{p = x * y\} & \leftarrow & \text{not in Presburger arithmetic} \\
 & & \textbf{while } p > 0 \textbf{ do } p := p - y \\
\{p = 0\} & & \\
 & & \Rightarrow \text{ valid but not provable}
\end{array}
$$

Definition: $A \subseteq \Sigma \to T$ is *expressive* iff. for every statement s and $Q \in A$

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$$wp\ s\ Q \in A$$

Corollary: If

- A is expressive,
- A contains *Bexp* (for all $b \in Bexp$, $\lambda\sigma.\mathcal{B}[b]\sigma \in A$), and
- A is closed under $\wedge$ and $\neg$ ($P, Q \in A$ then $\lambda\sigma.P\sigma \wedge Q\sigma \in A$)

then $\models \{P\}s\{Q\}$ with $P, Q \in A$ implies there is a proof of $\vdash \{P\}s\{Q\}$ containing only assertions in A.

Theorem: First-order arithmetic (FOA, Peano Arithmetic), i.e. FOL + $(\mathbb{N}, +, *, <)$, is expressive.

Proof: (ideas)

Gödel

1. Program state can be encoded as a natural number.
2. Program execution can be encoded as a FOA formula. For all s there is a formula $P_s(\sigma, \sigma')$ in FOA such that $P_s(\sigma, \sigma') \iff \langle s, \sigma \rangle \to \sigma'$
(Beware: Definition of $\langle \ldots,\ \ldots \rangle \to \ldots$ is not in FOA, because leastness of the relation is not expressible in FOA.)

$$
\begin{array}{rcl}
w & = & \textbf{while } b \textbf{ do } s \\
P_w(\sigma, \sigma') & = & \exists \text{ a sequence of states } \overline{\sigma} \text{ of length n .} \\
 & & \overline{\sigma_0} = \sigma \wedge \overline{\sigma_n} = \sigma' \wedge \neg\mathcal{B}[b]\sigma' \wedge \forall i < n.\mathcal{B}[b]\sigma_i \wedge P_s(\overline{\sigma_i}, \overline{\sigma_{i+1}})
\end{array}
$$

Question:

- Presburger Arithmetic contains $+$ but not $*$.
- Programs can synthesize $*$ from $+$.

$\Rightarrow$ Presburger Arithmetic is not expressive.

- FOA contains $+,*$ but not exponentiation.
- Programs can synthesize exponentiation from $*$.
- **Why is FOA expressive?**

Answer: Because there is a formula $E(n,m,e)$ such that $E(n,m,e) \iff e = n^m$

5.6 Relative Completeness

Intuition: finite proof system $\approx$ Prolog program

$\Rightarrow$ Main property of a finite proof system is that all provable formulae are recursively enumerable[19] (r.e.)

Theorem: If $\vdash$ is sound and recursively enumerable then it is not complete.

Proof: Remember: $\models \{ \textbf{ true } \} s \{ \textbf{ false } \}$ iff. s never terminates.

Theorem: $\{s \mid s \text{ never terminates}\}$ is not recursively enumerable.

(**Lemma:** $\{s \mid s \text{ terminates somewhere}\}$ *is* recursively enumerable.)

Assume $\vdash$ is sound, complete and recursively enumerable

$$\begin{array}{rcl} \vdash \text{ r.e.} & \Rightarrow & \{s \mid \ \vdash \{\textbf{true }\} s \{\textbf{false }\}\} \text{ is r.e.} \\ \vdash sound\&compl. & \Rightarrow & \{s \mid \ \models \{\textbf{true }\} s \{\textbf{false }\}\} \text{ is r.e.} \quad ↯ \quad qed. \end{array}$$

Thus our proof system for Hoare Logic cannot be recursively enumerable
Problem: $P \Rightarrow P'$ and $Q' \Rightarrow Q$ in consequence rule refer to truth, but not to some proof system.

Theorem: (Gödel)
The true formulae of FOA are not recursively enumerable, i.e. there is no finite, sound and complete proof system for FOA.

Choice in consequence rule: $P \Rightarrow P'$ and $Q' \Rightarrow Q$ refer to

1. truth in FOA. Then $\vdash$ is not recursively enumerable
2. provability in some standard proof system for FOA. Then $\vdash$ is incomplete.

Thus Hoare Logic is *relatively* complete with respect to (in-)completeness of the assertion logic.

[19]"rekursiv aufzählbar"

5.7 Verification condition generation

Extended syntax: $w = \textbf{while}\ b\ \textbf{do}\ \{I\}s$
Function *pre* is very similar to *wp* (weakest precondition) with exception of the new syntax for the while statement

$$
\begin{array}{rcl}
pre : Stm \to Assn & \to & Assn \\
pre\ \textbf{skip}\ Q & = & Q \\
pre\ (x := a)\ Q & = & Q[a/x] \\
pre\ (s_1;s_2)\ Q & = & pre\ s_1\ (pre\ s_2\ Q) \\
pre\ (\textbf{if}\ \ldots)\ Q & = & (b \Rightarrow pre\ s_1\ Q) \wedge \\
 & & (\neg b \Rightarrow pre\ s_2\ Q) \\
pre\ w\ Q & = & I
\end{array}
$$

As is clear, not all annotated partial correctness assertions are valid. To be so it is sufficient to establish the validity of certain assertions, called *verification conditions* for which all mention of commands is eliminated.

from Winskel book

$$
\begin{array}{rcl}
vc : Stm \to Assn & \to & Assn \quad vc : \text{verification condition} \\
 & & (\text{Soundness: } vc\ s\ Q \Rightarrow\ \vdash \{pre\ s\ Q\}s\{Q\}) \\
vc\ \textbf{skip}\ Q & = & \textbf{true} \\
vc\ (x := A)\ Q & = & \textbf{true} \\
vc\ (s_1;s_2)\ Q & = & vc\ s_1\ (pre\ s_2\ Q) \wedge vc\ s_2\ Q \\
vc\ (\textbf{if}\ \ldots)\ Q & = & vc\ s_1\ Q \wedge vc\ s_2\ Q \\
vc\ w\ Q & = & \underbrace{(I \wedge b \Rightarrow pre\ s\ I)}_{\text{"}I\text{ is invariant"}} \wedge (I \wedge \neg b \Rightarrow Q) \wedge (vc\ s\ I)
\end{array}
$$

Example:

$$
\begin{array}{rcl}
w & = & \textbf{while}\ I \neq N+1\ \textbf{do}\ \{P\}\ s \\
s & = & S := S+I;\ I := I+1 \\
P & = & (S = \sum_{i=1}^{I-1} i) \\
Q & = & (S = \sum_{i=1}^{N} i) \\
pre\ w\ Q & = & P \quad (P \text{ true e.g. if } I = 1 \text{ and } S = 0) \\
vc\ w\ Q & = & (P \wedge I \neq N+1 \Rightarrow pre\ s\ P) \wedge \\
 & & (P \wedge I = N+1 \Rightarrow Q) \wedge \\
 & & vc\ s\ P \\
vc\ s\ P & = & \textbf{true}\ \wedge \textbf{true} \quad \checkmark \\
pre\ s\ P & = & pre\ (S := S+I)\ (pre\ (I := I+1)\ P) \\
 & = & (S = \sum_{i=1}^{I} i) \\
 & = & \underbrace{S + I = \sum_{i=1}^{I} i}_{=P}
\end{array}
$$

Theorem: $vc\ s\ B \Rightarrow \vdash \{pre\ s\ B\}s\{B\}$

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Proof: by induction on s.

- $s = \textbf{skip}$: $vc\ \textbf{skip}\ B = \textbf{true} \Rightarrow \vdash \{\overbrace{pre\ \textbf{skip}\ B}^{B}\}\textbf{skip}\ \{B\}$
- $s = s_1;s_2$: $vc\ (s_1;s_2)\ B = vc\ s_1\ (pre\ s_2\ B) \wedge vc\ s_2\ B$
  $\overset{IH}{\Rightarrow} \vdash \{\underbrace{pre\ s_1\ (pre\ s_2\ B)}_{=pre\ (s_1;s_2)\ B}\}s_1\{pre\ s_2\ B\}$ and $\vdash \{pre\ s_2\ B\}s_2\{B\}$
  $\overset{HL}{\Rightarrow} \vdash \{pre\ (s_1;s_2)\ B\}s_1;s_2\{B\}$
- $s = \textbf{if}\ b\ \textbf{then}\ s_1\ \textbf{else}\ s_2$: $vc\ s\ B = vc\ s_1\ B \wedge vc\ s_2\ B$
  $\overset{IH}{\Rightarrow} \vdash \{pre\ s_i\ B\}s_i\{B\} \overset{conseq.}{\Rightarrow} \vdash \{b \wedge pre\ s_1\ B\}s_1\{B\} \wedge \vdash \{\neg b \wedge pre\ s_2\ B\}s_2\{B\}$
  precondition strengthening

  $$\left(\text{if-then-else-rule}\ \frac{\vdash \{b \wedge pre\ s_1\ B\}s_1\{B\} \qquad \vdash \{\neg b \wedge pre\ s_2\ B\}s_2\{B\}}{\vdash \{pre\ s\ B\}s\{B\}}\right)$$

- $s = \textbf{while}\ b\ \textbf{do}\ \{I\}s_0$
  $vc\ s\ B = (I \wedge \neg b \Rightarrow B) \wedge (I \wedge b \Rightarrow pre\ s_0\ I) \wedge vc\ s_0\ I$

$$\text{conseq}\ \frac{I \Rightarrow I \qquad \text{w-rule}\ \dfrac{\text{conseq}\ \dfrac{I \wedge b \Rightarrow pre\ s_0\ I \qquad \dfrac{vc\ s_0\ I \wedge I.H.}{\vdash \{pre\ s_0\ I\}s_0\{I\}} \qquad I \Rightarrow I}{\vdash \{I \wedge b\}s_0\{I\}}}{\vdash \{I \wedge \neg b\}w\{I \wedge \neg b\}} \qquad I \wedge \neg b \Rightarrow B}{\vdash \{\underbrace{pre\ w\ B}_{=I}\}w\{B\}}$$

Corollary: $\left\{ \begin{array}{l} vc\ s\ B \wedge \\ A \Rightarrow pre\ s\ B \end{array} \right\} \Rightarrow \vdash \{A\}s\{B\}$

! Only helpful if *pre* and *vc* "sensible".

- $pre\ s\ B = \textbf{false}$ not interesting ($\{\ \textbf{false}\ \}s\{Q\}$ always provable)
- $vc\ s\ B = \textbf{false}$ not interesting

Needed: completeness!

Theorem: If $\vdash \{A\}s\{B\}$ (s is not annotated), then there is an annotated version s' of s such that $vc\ s'\ B$ and $A \Rightarrow pre\ s'\ B$.

Proof: by rule induction on $\vdash \{A\}s\{B\}$

- Case w-rule:

  $$\frac{\dfrac{\vdots}{\vdash \{A \wedge b\}s\{A\}}}{\vdash \{A\}w\{\underbrace{A \wedge \neg b}_{B}\}} \qquad (w = \textbf{while}\ b\ \textbf{do}\ s)$$

  $IH \Rightarrow \exists$ an annotated version s' of s such that $vc\ s'\ A$ and $A \wedge b \Rightarrow pre\ s'\ A$

  $w' = \textbf{while}\ b\ \textbf{do}\ \{A\}s \quad (\Rightarrow (A \Rightarrow pre\ w'\ B))$
  $vc\ w'\ B = (A \wedge \neg b \Rightarrow B) \wedge (A \wedge b \Rightarrow pre\ s'\ A) \wedge vc\ s'\ A$

- Case conseq. rule:

  $$\frac{A \Rightarrow A' \quad \vdash \{A'\}s\{B'\} \quad B' \Rightarrow B}{\vdash \{A\}s\{B\}}$$

  IH : $\exists$ annotated version of s' of s such that $\underbrace{vc\ s'\ B'}_{\Rightarrow vc\ s'\ B}$ and $\underbrace{A' \Rightarrow pre\ s'\ B'}_{\Uparrow\ \Downarrow\ A \Rightarrow pre\ s'\ B}$ by lemma:

  Lemma: If $P \Rightarrow Q$ then $vc\ s\ P \Rightarrow vc\ s\ Q$ and $pre\ s\ p \Rightarrow pre\ s\ Q$.

5.8 Total Correctness

Total correctness = partial correctness $\wedge$ termination

$\models_t \{A\}s\{B\}$ iff. $\models \{A\}s\{B\}$ and

1. $\forall \sigma. A\sigma \Rightarrow \exists \sigma'. \langle s,\sigma\rangle \rightarrow \sigma'$
   only for deterministic languages, or
2. $(\langle s,\sigma\rangle \rightarrow_1 \langle s',\sigma'\rangle$: small-step semantics)
   there is no infinite chain $\langle s,\sigma\rangle \rightarrow_1 \langle s_1,\sigma_1\rangle \rightarrow_1 \langle s_2,\sigma_2\rangle \rightarrow_1 \ldots$ such that $A\sigma$ holds, or
3. $\forall \sigma. A\sigma \Rightarrow s \downarrow \sigma$ ("s started in σ *must* terminate")
   (analogous to $\sigma \in T[s]$)

   Inductive definition of $\downarrow$:

   - **skip** $\downarrow \sigma$
   - $(x := a) \downarrow \sigma$
   - $\dfrac{s_1 \downarrow \sigma \quad \forall \sigma' \langle s,\sigma\rangle \rightarrow \sigma' \Rightarrow s_2 \downarrow \sigma'}{s_1; s_2 \downarrow \sigma}$
   - $\dfrac{s_1 \downarrow \sigma \quad \mathcal{B}[b]\sigma}{\textbf{if}\ b\ \ldots \downarrow \sigma} \quad \dfrac{s_2 \downarrow \sigma \quad \neg\mathcal{B}[b]\sigma}{\textbf{if}\ b\ \ldots \downarrow \sigma}$
   - $\dfrac{\neg\mathcal{B}[b]\sigma}{w \downarrow \sigma} \quad \dfrac{\mathcal{B}[b]\sigma \quad (s;w) \downarrow \sigma}{w \downarrow \sigma}$

Proof rules for total correctness:

$$\frac{\{A \wedge b \wedge t = z\}s\{A \wedge t < z\} \quad A \Rightarrow t \geq 0}{\vdash \{A\}w\{A \wedge \neg b\}} \tag{8}$$

where $t \in Aexp$ and z is a "new" variable, i.e. z does not occur in A, b, t.

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This rule is *seriously incomplete* if $t \in Aexp$ because $Aexp$ can only express polynomials, t is an upper bound on the running time of the loops, but we can program loops with super-polynomial running time.

Alternative: extend set of functions allowed to occur in t.
Better: new proof rule without explicit t:

$$\frac{\vdash \{A(n+1)\}s\{A(n)\}}{\vdash \{\exists n \in \mathbb{N}. A(n)\}w\{A(0)\}} \tag{9}$$

if $A(0) \Rightarrow \neg b$ and $A(n+1) \Rightarrow b$ (side conditions).
This rule is sound and in a certain sense complete (for first order arithmetic).

Example: for (8)

$$\frac{\{A \land N = 0 \land N = Z\}\ N := N-1;\ M := M+1\ \{A \land N < Z\} \qquad A \Rightarrow N \geq 0}{\vdash \{\underbrace{N \geq 0 \land N + M = C}_{A}\}\ \textbf{while}\ N \neq 0\ \textbf{do}\ (N := N-1;\ M := M+1)\ \{A \land N = 0\}}$$

Example: for (9)

$$\text{conseq}\frac{\dfrac{\vdash \{x = n+1\} x := x-1 \{x = n\}}{\vdash \{\exists n. x = n\} w \{x = 0\}}}{\vdash \{x \geq 0\} \textbf{while}\ x \neq 0\ \textbf{do}\ x := x-1 \{x = 0\}}$$

side conditions: $x = 0 \Rightarrow \neg(x \neq 0)$, $x = n+1 \Rightarrow x \neq 0$.

Example:

$$\text{conseq}\frac{\dfrac{\dfrac{x + (n+1)*k \geq a > x + n*k \Rightarrow (x+k) + n*k \geq a > (x+k) + (n-1)*k}{\vdash \{x + (n+1)*k \geq a > x + n*k\}\ x := x+k\ \{x + n*k \geq a > x + (n-1)*k\}}}{\vdash \{\exists n. x + n*k \geq a > x + (n-1)*k\}\ w\ \{x + 0*k \geq a \geq x - k\}}}{\vdash \{\ \textbf{true}\ \}\ \textbf{while}\ x < a\ \textbf{do}\ x := x + \underbrace{k}_{\in \mathbb{N}}\ \{x \geq a\}}$$

side conditions: $x \geq a \Rightarrow \neg x < a$ and $x + (n+1)*k \geq a > x + n*k \Rightarrow x < a$

Remark: $S_1\{N = "f(M)"\}$ **while** $X < N +$ **do** $X := X + 1$
there is a predicate F such that $N = f(M)$ iff. $F(M, N)$.

$$\frac{}{\{\exists n. f(M) - X = n\}\ w\ \{X = N\}}$$

$\{\exists n. F(M, N) \land N - X = n\}$

5.9 Extension of WHILE

5.9.1 Nondeterminism

s_1 **or** s_2

$$\frac{\vdash \{P\}\ s_1\ \{Q\} \qquad \vdash \{P\}\ s_2\ \{Q\}}{\vdash \{P\}\ s_1\ \textbf{or}\ s_2\ \{Q\}}$$

```
while X != 0 do
  if X < 0 then X := some arbitrary positive number
           else X := X-1
```

terminates, but number of iterations is not a function of X, is unbounded.
Termination measure: $(X < 0, X)$
Ordering:

$$\begin{array}{rcl} (\textbf{true}, _) & > & (\textbf{false}, _) \\ (\textbf{false}, m) & > & (\textbf{false}, n) \text{ iff. } m > n \end{array}$$

Example product of two orderings.

5.9.2 Arrays

Extended syntax:

$$\begin{array}{rcl} Aexp & ::= & \ldots \mid A[a] \\ Stm & ::= & \ldots \mid A[a_1] := a_2 \end{array}$$

$A, A_1, A_2, \ldots$ are array variables.
$\mathcal{A}$ is the evaluation function.

Problem: assignment axiom not valid:

$$\begin{array}{lcl} \not\models \{0 = 0\} & A[A[1]] := 0 & \{A[A[1]] = 0\} \\ A[1] = 1 & & A[A[1]] = 42 \\ A[0] = 42 & & 0 \end{array}$$

Solution: regard arrays as *one* variable, not as a sequence of variables $A[1], A[2], \ldots$
$\Rightarrow$ Assignment must change A as a whole.

Model: array = $\mathbb{Z} \to \mathbb{Z}$
Translate **WHILE** + array $\mapsto$ **WHILE** + function variables

1. Replace $A[a_1] := a_2$ by $A := A[a_1 \mapsto a_2]$
2. replace $A[a]$ by $A(a)$

Result: **WHILE** program in an extended set of $Aexp$:

$$\begin{array}{rcl} Aexp & ::= & \ldots \mid f(a) \\ Funs & ::= & A_1 \mid A_2 \mid \ldots \mid f[a_1 \mapsto a_2] \quad f \in Funs \end{array}$$

Example: $A[A[1]] := 0$

$$\begin{array}{ll} \mapsto & A := A[A[1] \mapsto 0] \\ \mapsto & A := A[A(1) \mapsto 0] \\ & NB : A[_ \mapsto _] \text{ part of programming language syntax!} \end{array}$$

To prove $\vdash \{A[1] \neq 1\}\ A[A[1]] := 0\ \{A[A[1]] = 0\}$
we prove $\vdash \{A(1) \neq 1\}\ A := A[A(1) \mapsto 0]\ \{A(A(1)) = 0\}$
Need to show: $A(1) \neq 1 \Rightarrow A'(A'(1)) = 0$
Assume $A(1) \neq 1$

$$\begin{array}{rcl} A'(A'(1)) & \overset{Def.of A' \wedge A(1) \neq 1}{=} & A'(A(1)) \\ & \overset{Def.of A'}{=} & 0 \end{array}$$

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5.9.3 Procedures

1 procedure without parameters
New statement: **call**
Implicit procedure definition: **proc** = body
Operational semantics:

$$\frac{\langle body, \sigma\rangle \to \sigma'}{\langle \mathbf{call}\,, \sigma\rangle \to \sigma'}$$

New format of Hoare rules:

$$H \vdash \{P\}\ s\ \{Q\} \tag{10}$$

where H[20] is 0 or 1 Hoare triple (of the form $\{A\}$ **call** $\{B\}$).
(10) "$\{P\}\ s\ \{Q\}$ is provable under the assumption H"

Adaption of old proof rules to a new format: add $H \vdash$ everywhere:

$$\frac{P \Rightarrow P' \qquad H \vdash \{P'\}\ s\ \{Q'\} \qquad Q' \Rightarrow Q}{H \vdash \{P\}\ s\ \{Q\}}$$

New rule for **call**

$$\frac{\overbrace{\{P\}\ \mathbf{call}\ \{Q\}}^{P(n)} \vdash \overbrace{\{P\}\ body\ \{Q\}}^{P(n+1)}}{\vdash \{P\}\ \mathbf{call}\ \{Q\}}$$

Assumption rule: $\{P\}\ s\ \{Q\} \vdash \{P\}\ s\ \{Q\}$

Example: (1)

$$\text{call rule}\ \frac{\text{assumption rule}\ \dfrac{body = \mathbf{call}}{\{P\}\ \mathbf{call}\ \{Q\} \vdash \{P\}\ \mathbf{call}\ \{Q\}}}{\vdash \{P\}\ \mathbf{call}\ \{Q\}}$$

Example:

$$\begin{array}{llll} body = \mathbf{if}\ I = 0 & \mathbf{then} & F := 1 \\ & \mathbf{else} & \underbrace{(I := I - 1;\ \mathbf{call}\ ;\ I := I + 1;\ F := I * F)}_{s} \end{array}$$

$$\frac{\dfrac{H \vdash \{I = 0\}\ F := 1\ \{F = I!\} \qquad \dfrac{H \vdash \{I \neq 0\}\ I := I - 1\ \{\mathbf{true}\ \}\ \overset{H}{\mathbf{call}}\ \{F = I!\}\ I := I + 1\ \{F = (I-1)!\}\ F = I * F\ \{F = I!\}}{H \vdash \{I \neq 0\}\ s\ \{F = I!\}}}{H \vdash \{\mathbf{true}\ \}\ body\ \{F = I!\}}}{\vdash \{\mathbf{true}\ \}\ \mathbf{call}\ \{F = I!\}}$$

Example:

$$\begin{array}{lll} body = \mathbf{if}\ I = 0 & \mathbf{then} & \mathbf{skip} \\ & \mathbf{else} & (I := I - 1;\ \mathbf{call}\ I := I + 1) \end{array}$$

$$\frac{\dfrac{H \vdash \{I = i \wedge I = 0\}\ \mathbf{skip}\ \{I = i\} \qquad H \vdash \{I = i\}\ I := I - 1\ \{I = i - 1\}\ \overset{?}{\mathbf{call}}\ \{I = i - 1\}\ I := I + 1\ \{I = i\}}{H \vdash \{I = i\}\ body\ \{I = i\}}}{\vdash \underbrace{\{I = i\}\ \mathbf{call}\ \{I = i\}}_{H}}$$

[20] (Hypothesis)

NB: i does not occur in the program. i is an auxiliary variable – remembers pre-values of I. Need to adapt the value of auxiliary variables.

A specification of addition?

$$\{\ \mathbf{true}\ \}\ X := 0;\ Y := 0;\ Z := 0\ \{\ Z = X + Y\ \}$$

With auxiliary variables:

$$\{X = x \wedge Y = y\} \quad Z := X + Y \quad \{Z = x + y\}$$

where x and y are auxiliary variables.

⌈ Alternative:

$$\{\ \mathbf{true}\ \} \qquad \{\ Z = X_{pre} + Y_{pre}\ \} \quad ⌋$$

Assertions = Predicates of (State of program variables × state of auxiliary variables)
Convention: auxiliary variable (i) = copy of program variable (I)
⇒ use Σ for both program and auxiliary variables

$$\text{Formally: Assn} = \underbrace{\Sigma}_{progr.\ var.} \times \underbrace{\Sigma}_{aux.\ var.} \rightarrow T$$

Convention: τ represents state of auxiliary variables.

Problem:

$$?\frac{\{I = i\}\ \mathbf{call}\ \{I = i\}}{\{I = i-1\}\ \mathbf{call}\ \{I = i-1\}} \tag{11}$$

New consequence rule

$$\frac{H \vdash \{P'\}\ s\ \{Q'\}}{H \vdash \{P\}\ s\ \{Q\}} \text{ if } \forall\sigma,\sigma'.\overbrace{\big(\forall\tau.P'(\sigma,\tau) \Rightarrow Q'(\sigma',\tau)\big)}^{\approx\models\{P'\}\ s\ \{Q\}} \Rightarrow \overbrace{\big(\forall\tau.P(\sigma,\tau) \Rightarrow Q(\sigma',\tau)\big)}^{\approx\models\{P\}\ s\ \{Q\}}$$

τ is universally quantified!

Validity with auxiliary variables:
$\models \{P\}\ s\ \{Q\} \equiv \forall\sigma,\sigma'.\langle s,\sigma\rangle \rightarrow \sigma' \Rightarrow \big(\forall\tau.P(\sigma,\tau) \Rightarrow Q(\sigma',\tau)\big)$

Solution for (11) by new consequence rule because

$$\forall \underbrace{I}_{\sigma}, \underbrace{I'}_{\sigma'} . \underbrace{(\forall \underbrace{i}_{\tau} . I = i \rightarrow I' = i)}_{\text{instantiate with } i-1} \Rightarrow (\forall i. I = i-1 \Rightarrow I' = i-1) \quad \checkmark$$

Old consequence rule is a special case of the new one:
Assume old side conditions:

1. $\forall\sigma,\tau.P(\sigma,\tau) \Rightarrow P'(\sigma,\tau)$
2. $\forall\sigma,\tau.Q'(\sigma,\tau) \Rightarrow Q(\sigma,\tau)$

Prove side condition of new consequence rule:
Assume (3) $\forall\tau.P'(\sigma,\tau) \Rightarrow Q'(\sigma',\tau)$.
Prove $\forall\tau.P(\sigma,\tau) \Rightarrow Q(\sigma',\tau)$

$$P(\sigma,\tau) \overset{(1)}{\Rightarrow} P'(\sigma,\tau) \overset{(3)}{\Rightarrow} Q'(\sigma',\tau) \overset{(2)}{\Rightarrow} Q(\sigma',\tau)$$

Soundness of new rule:
Assume $\models \{P'\}\ s\ \{Q'\} \Leftrightarrow \forall \sigma, \sigma'. \langle s, \sigma \rangle \rightarrow \sigma' \Rightarrow \underbrace{(\forall \tau. P'(\sigma, \tau) \Rightarrow Q'(\sigma', \tau))}_{(B)}$

Prove $\models \{P\}\ s\ \{Q\}$
Assume $\langle s, \sigma \rangle \rightarrow \sigma' \quad (A)$
Prove $\underbrace{\forall \tau. P(\sigma, \tau) \Rightarrow Q(\sigma', \tau)}_{(C)}$
$(A) \Rightarrow (B) \Rightarrow (C)$

Soundness

Validity of triples under assumptions:
$T_1 \models T_2 \equiv (\models T_1 \Rightarrow \models T_2)$

Theorem: $H \vdash \{P\}\ s\ \{Q\} \Rightarrow H \models \{P\}\ s\ \{Q\}$

Proof: "Validity is preserved by every rule."

- for basic constructs: as usual
- for new consequence rule: see above
- for assumption rule: ✓
- for call rule: $\dfrac{\{P\}\ \textbf{call}\ \{Q\} \vdash \{P\}\ body\ \{Q\}}{\vdash \{P\}\ \textbf{call}\ \{Q\}}$

  01/30/2002

  IH: $\{P\}\ \textbf{call}\ \{Q\} \models \{P\}\ body\ \{Q\}$
  i.e. $\models \{P\}\ \textbf{call}\ \{Q\} \rightarrow \models \{P\}\ body\ \{Q\}$

  Goal: $\{P\}\ \textbf{call}\ \{Q\}$ iff. $\{P\}\ body\ \{Q\}$
  (by definition of $\models$ and operational semantics)

???

Finer grained operational semantics:
$\langle s, \sigma \rangle \rightarrow_n \sigma'$ where n indicates the maximum recursion depth.

$\vec{n}$

Some of the rules:

$$\frac{\langle body, \sigma \rangle \rightarrow_n \sigma'}{\langle \textbf{call}\ , \sigma \rangle \rightarrow_{n+1} \sigma'}$$

$$\frac{\langle s_1, \sigma_1 \rangle \rightarrow_n \sigma_2 \qquad \langle s_2, \sigma_2 \rangle \rightarrow_n \sigma_3}{\langle s_1; s_2, \sigma_1 \rangle \rightarrow_n \sigma_3}$$

$$\models_n \{P\}\ s\ \{Q\} \equiv \forall \sigma, \sigma'. \langle s, \sigma \rangle \rightarrow_n \sigma' \Rightarrow (\forall \tau. P(\sigma, \tau) \Rightarrow Q(\sigma', \tau))$$
$$T_1 \models_n T_2 \equiv \models_n T \Rightarrow \models_n T_2$$

Theorem: $C \vdash \{P\}\ s\ \{Q\} \Rightarrow \underbrace{\forall n. C \models_n \{P\}\ s\ \{Q\}}$

Corollary (Proof needs theorems about the relationship of $\langle s, \sigma \rangle \rightarrow\rightarrow \sigma'$ and $\langle s, \sigma \rangle \rightarrow_n \sigma$

Proof: by induction on $C \models \{P\}\ s\ \{Q\}$
call : IH: $\forall n.\{P\}$ **call** $\{Q\} \models_n \{P\}\ body\ \{Q\}$
i.e. $\models_n \{P\}$ **call** $\{Q\} \rightarrow \models_n \{P\}\ body\ \{Q\}$
Goal: $\forall n. \models_n \{P\}$ **call** $\{Q\}$
Proof by induction on n:

1. $n = 0$ ✓
   ($\models_0 \{P\}$ **call** $\{Q\}$ is always true because $\langle \textbf{call}, \sigma\rangle \rightarrow_0 \sigma'$ is always false)

2. $n \rightsquigarrow n+1$

$$\begin{array}{rcl} IH_2 & \Rightarrow & \models_n \{P\}\ \textbf{call}\ \{Q\} \\ IH_1 & \Rightarrow & \models_n \{P\}\ body\ \{Q\} \\ & \Rightarrow & \models_{n+1} \{P\}\ \textbf{call}\ \{Q\} \end{array}$$

Completeness

$MGT(s)$[21] $= \{\lambda\sigma\tau.\tau = \sigma\}\ s\ \{\lambda\sigma'\tau.\langle s,\tau\rangle \rightarrow \sigma'\}$

$\models MGT(s) : \langle s,\sigma\rangle \rightarrow \sigma' \Rightarrow \underbrace{(\forall\tau.\tau = 0 \Rightarrow \langle s,\tau\rangle \rightarrow \sigma')}_{\langle s,\sigma\rangle \rightarrow \sigma'}$

Lemma: $\vdash MGT\ s$ (for all s) implies completeness.

Proof: $\dfrac{\vdash MGT\ s}{\vdash \{P\}\ s\ \{Q\}}$ because $\overbrace{(\forall\tau.\tau = \sigma \Rightarrow \langle s,\tau\rangle \rightarrow \sigma')}^{\langle s,\sigma\rangle \rightarrow \sigma'} \Rightarrow (\forall\tau.P(\sigma,\tau) \Rightarrow Q(\sigma',\tau))$

$\models \{P\}\ s\ \{Q\} \equiv \forall\sigma,\sigma'.\langle s,\sigma\rangle \rightarrow \sigma' \Rightarrow (\forall\tau.P(\sigma,\tau) \Rightarrow Q(\sigma',\tau))$
i.e. $\models \{P\}\ s\ \{Q\}$ implies side condition of new consequence rule.

Lemma: (1) $H \vdash MGT$ **call** $\Rightarrow H \vdash MGT\ s$

Now:

$$\begin{array}{rcl} & MGT\ \textbf{call} \vdash MGT\ \textbf{call} & \text{by assumption rule} \\ \Rightarrow & MGT\ \textbf{call} \vdash MGT\ body & \text{by Lemma (1)} \\ \Rightarrow & \vdash MGT\ \textbf{call} & \text{by } \textbf{call} \text{ rule} \\ \Rightarrow & \vdash MGT\ s & \text{by Lemma (1)} \end{array}$$

Proof: of Lemma (1) by induction on s.

- $s =$ **call** : ✓
- $s =$ **while** b **do** s_0

$$\uparrow \text{conseq.} \frac{H \vdash \{\lambda\sigma\tau.\tau = \sigma\}\ s\ \{\lambda\sigma'\tau.\langle s,\tau\rangle \rightarrow \sigma'\}}{H \vdash \{\lambda\sigma\tau.I(\sigma,\tau)\}\ s\ \{\lambda\sigma'\tau.I(\sigma',\tau) \wedge \neg\mathcal{B}[b]\sigma'\}}$$

$$I(\sigma,\tau) = (\tau,\sigma) \in \{(\sigma,\sigma') \mid \langle s_0,\sigma\rangle \rightarrow \sigma' \wedge \mathcal{B}[b]\sigma\}^*$$

[21] **M**ost **G**eneral **T**riple

= a finite number of iterations of s_0 take τ to σ. Side condition provable by induction on $*$.

5.10 A Hoare Logic for Time

Aim: $\{A\}\ s\ \{e \Downarrow B\} \approx$ execution of s takes time $O(e)$.

Plan:

1. operational semantics with exact time
2. timed triples
3. Hoare Logic for timed triples
4. Examples
5. Soundness

5.10.1 Operational Semantics with time

$$\mathcal{TA} : Aexp \to \mathbb{N}$$

Time for evaluating an $Aexp$. Machine model: unit cost for all arithmetic operations.

Definition of $\mathcal{TA}$:

$$\begin{aligned} & \mathcal{TA}(a_1\ aop\ a_2) = 1 + \mathcal{TA}(a_1) + \mathcal{TA}(a_2) \\ & \dots \\ \Rightarrow\ & \mathcal{TA}(a) = \text{size of } a \text{ (as a tree)} \end{aligned}$$

Similarly: $\mathcal{TB} : Bexp \to \mathbb{N}$

For statements: $\langle s, \sigma \rangle \to^n \sigma'$

$$\begin{array}{lll} \langle \mathbf{skip}\ , \sigma \rangle & \to^1 & \sigma \\ \langle x := a, \sigma \rangle & \to^{\mathcal{TA}(a)} & \sigma[x \mapsto \mathcal{A}[a]\sigma] \end{array}$$

$$\frac{\langle s_1, \sigma_1 \rangle \to^{n_1} \sigma_2 \qquad \langle s_2, \sigma_2 \rangle \to^{n_2} \sigma_3}{\langle s_1 ; s_2, \sigma \rangle \to^{n_1+n_2} \sigma_3}$$

$$\frac{\mathcal{B}[b]\sigma = \text{tt} \qquad \langle s_1, \sigma \rangle \to^n \sigma'}{\langle \mathbf{if}\ b\ \mathbf{then}\ s_1, \sigma \rangle \to^{n+\mathcal{TB}(b)} \sigma'}$$

$$\frac{\mathcal{B}[b]\sigma = \text{ff}}{\langle w, \sigma \rangle \to^{\mathcal{TB}(b)} \sigma} \qquad \frac{\mathcal{B}[b]\sigma = \text{tt} \qquad \langle s, \sigma \rangle \to^m \sigma' \qquad \langle w, \sigma' \rangle \to^n \sigma''}{\langle w, \sigma \rangle \to^{m+n+\mathcal{TB}(b)} \sigma''}$$

5.10.2 Timed Hoare triples

$\models \{A\}\ s\ \{e \Downarrow B\} \equiv$ there is some $k > 0$ ($k \in \mathbb{N}$ or $\mathbb{R}$) such that for all σ: if $A(\sigma)$ then there is some σ' and some n such that $\langle s, \sigma\rangle \longrightarrow^n \sigma'$ and $n \leq k * \mathcal{A}[e]\sigma$ and $B(\sigma')$.

NB: e a function of the *initial* state.

02/04/2002

Q: How to prove $\vdash \{I = J\}$ **call** $\{I = J\}$ where $body =$ **if** $I = 0$ **then skip else** $(I := I - 1;$ **call** $; I := I + 1)$

1. Prove $\text{conseq}\dfrac{\vdash \{I = i \wedge J = j\}\ \textbf{call}\ \{I = i \wedge J = j\}}{\vdash \{I = J\}\ \textbf{call}\ \{I = J\}}$

   $(\forall i, j. I = i \wedge J = j \Rightarrow I' = i \wedge J' = j)$ (set $i = j$)
   $\Rightarrow (\forall i, j. I = J \Rightarrow I' = J')$

Hoare Logic with time:

$$\begin{array}{lcl} \models \{X = 42\} & \textbf{while } X \neq 0 \textbf{ do } X := X - 1 & \{\overset{1}{X} \Downarrow X = 0\} \\ \models \{X \geq 0\} & '' & \{X \Downarrow X = 0\} \\ \models \{\textbf{ true }\} & '' & \text{no possible assertion} \end{array}$$

Finer grained definition of $\models$ with separated auxiliary variables (τ)

$$\models \{A\}\ s\ \{t \Downarrow B\} \equiv \exists k > 0. \forall \sigma, \tau. A(\sigma, \tau) \Rightarrow \exists \sigma', n. \langle s, \sigma\rangle \longrightarrow^n \sigma' \wedge n \leq k * \mathcal{A}[t](\sigma, \tau) \wedge B(\sigma', \tau)$$

5.10.3 Rules of Hoare Logic for Time

$$\begin{array}{ccc} \{B[a/x]\} & x := a & \{1 \Downarrow B\} \\ \{A\} & \textbf{skip} & \{1 \Downarrow A\} \end{array}$$

$$\frac{\{A_1\}\ s_1\ \{t_1 \Downarrow A_2\} \qquad \{A_2\}\ s_2\ \{t_2 \Downarrow A_3\}}{\{A_1\}\ s_1; s_2\ \{t_1 + t_2 \Downarrow A_3\}}$$

Example:

$$\lightning\ \frac{\{Y \geq 0\}\ X := Y\ \{1 \Downarrow X \geq 0\} \qquad \{X \geq 0\}\ w\ \{X \Downarrow X = 0\}}{\{Y \geq 0\}\ X := Y;\ \underbrace{\textbf{while } X \geq 0 \textbf{ do } X := X - 1}_{w}\ \{1 + X \Downarrow X = 0\}}$$

Problem: t_2 refers to the intermediate state, not to the initial state.

New rule:

$$\frac{\{A_1 \wedge t_2' = u\}\ s_1\ \{t_1 \Downarrow A_2 \wedge t_2 \leq u\} \qquad \{A_2\}\ s_2\ \{t_2 \Downarrow A_3\}}{\{A_1\}\ s_1; s_2\ \{t_1 + t_2 \Downarrow A3\}}$$

where u is some new auxiliary variable.

Example:

$$\frac{\{Y \geq 0 \wedge \overset{t_2'}{Y} = u\}\ X := Y\ \{\overset{t_1}{1} \Downarrow X \geq 0 \wedge \overset{t_2}{X} \leq u\} \qquad \{X \geq 0\}\ w\ \{\overset{t_2}{X} \Downarrow X \geq 0\}}{\{Y \geq 0\}\ X := Y;\ \underbrace{\textbf{while } X \geq 0 \textbf{ do } X := X - 1}_{w}\ \{1 + Y \Downarrow X \geq 0\}}$$

if -rule:

$$\frac{\{A \wedge b\}\ s_1\ \{t \Downarrow B\} \qquad \{A \wedge \neg B\}\ s_2\ \{t \Downarrow B\}}{\{A\}\ \textbf{if}\ b\ \textbf{then}\ \ldots\ \{t \Downarrow B\}}$$

If s_1 and s_2 have different complexities: use consequence rule:

$$\frac{A \Rightarrow (A' \wedge \exists k > 0.t' \leq k \cdot t) \qquad \{A'\}\ s\ \{t' \Downarrow B'\} \qquad B' \Rightarrow B}{\{A\}\ s\ \{t \Downarrow B\}}$$

Example:

$$\text{conseq}\,\frac{\text{conseq}\,\dfrac{\{A\}\ s\ \{1 \Downarrow B\}}{\{A\}\ s\ \{2 \Downarrow B\}}}{\{A\}\ s\ \{1 \Downarrow B\}}$$

$$\frac{\{X \geq 0\}\ s\ \{1 \Downarrow B\}}{\{''\}\ s + \{X \Downarrow B\}}$$

$$\frac{\{A(n+1) \wedge t' = u\}\ s\ \{t_s \Downarrow A(n) \wedge t \leq u\}}{\{\exists n.A(n)\}\ w\ \{t \Downarrow A(0)\}}$$

$$if \qquad \begin{array}{c} A(0) \Rightarrow \neg b \wedge t \geq 1 \\ A(n+1) \Rightarrow b \wedge t \geq t' + t_s \end{array}$$

$$\frac{\{\ldots \wedge t' = u\}\ s\ \{t_s \Downarrow \ldots \wedge t \leq u\} \qquad \{\ldots\}\ w\ \{t \Downarrow \ldots\}}{\{\ldots\}\ s;w\ \{t' + t_s \Downarrow \ldots\}}$$

Example:

$$\text{cnsq}\,\frac{\dfrac{\text{cnsq}\,\dfrac{\dfrac{\{\overbrace{I = n+1 \wedge I = u}^{A_2} \wedge 1 = v\}\ S := S+I\ \{1 \Downarrow A_2 \wedge 1 \leq v\} \qquad \{A_2\}\ I := I-1\ \{1 \Downarrow I = n \wedge I+1 \leq u\}}{\{I = n+1 \wedge \overbrace{I}^{t'} = u\}\ S := S+I;\ I := I-1\ \{2 \Downarrow I = n \wedge I+1 \leq u\}}}{\{I = n+1 \wedge \overbrace{I}^{t'} = u\}\ S := S+I;\ I := I-1\ \{1 \Downarrow I = n \wedge I+1 \leq u\}}}{\{\exists n.I = n\}\ w\ \{I+1 \Downarrow \textbf{true}\}}}{\{I \geq 0\}\ \textbf{while}\ I \neq 0\ \textbf{do}\ (S := S+I;\ I := I-1)\ \{\underbrace{I+1}_{t} \Downarrow \textbf{true}\}}$$

$$\begin{array}{c} I = 0 \Rightarrow \neg(I \neq 0) \wedge I+1 \geq 1 \\ I = n+1 \Rightarrow I \neq 0 \wedge I+1 \geq I+1 \quad \checkmark \end{array}$$

$\mathcal{EOF}$

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