Concrete Semantics

A Proof Assistant Approach

Tobias Nipkow

Fakultät für Informatik
Technische Universität München

2014-1-26

Part I

Isabelle

Chapter 2

Programming and Proving

1. Overview of Isabelle/HOL
2. Type and function definitions
3. Induction Heuristics
4. Simplification

Notation

Implication associates to the right:

$$A \Longrightarrow B \Longrightarrow C \quad \text{means} \quad A \Longrightarrow (B \Longrightarrow C)$$

Similarly for other arrows: $\Rightarrow$, $\longrightarrow$

$$\frac{A_1 \quad \ldots \quad A_n}{B} \quad \text{means} \quad A_1 \Longrightarrow \ldots \Longrightarrow A_n \Longrightarrow B$$

1. Overview of Isabelle/HOL
2. Type and function definitions
3. Induction Heuristics
4. Simplification

HOL = Higher-Order Logic

HOL = Functional Programming + Logic

HOL has

- datatypes
- recursive functions
- logical operators

HOL is a programming language!

Higher-order = functions are values, too!

HOL Formulas:

- For the moment: only $term = term$,
  e.g. $1 + 2 = 4$
- Later: $\wedge$, $\vee$, $\longrightarrow$, $\forall$, ...

Types

Basic syntax:

$$
\begin{array}{rcll}
\tau & ::= & (\tau) & \\
& | & bool \;|\; nat \;|\; int \;|\; \ldots & \text{base types} \\
& | & 'a \;|\; 'b \;|\; \ldots & \text{type variables} \\
& | & \tau \Rightarrow \tau & \text{functions} \\
& | & \tau \times \tau & \text{pairs (ascii: *)} \\
& | & \tau\ list & \text{lists} \\
& | & \tau\ set & \text{sets} \\
& | & \ldots & \text{user-defined types}
\end{array}
$$

Convention: $\tau_1 \Rightarrow \tau_2 \Rightarrow \tau_3 \equiv \tau_1 \Rightarrow (\tau_2 \Rightarrow \tau_3)$

Terms

Terms can be formed as follows:

- *Function application:*
  $f\ t$
  is the call of function f with argument t.
  If f has more arguments: $f\ t_1\ t_2\ \ldots$
  Examples: $sin\ \pi$, $plus\ x\ y$
- *Function abstraction:*
  $\lambda x.\ t$
  is the function with parameter x and result t,
  i.e. "$x \mapsto t$".
  Example: $\lambda x.\ plus\ x\ x$

Terms

Basic syntax:

$$
\begin{array}{rcll}
t & ::= & (t) & \\
& | & a & \text{constant or variable (identifier)} \\
& | & t\ t & \text{function application} \\
& | & \lambda x.\ t & \text{function abstraction} \\
& | & \ldots & \text{lots of syntactic sugar}
\end{array}
$$

Examples: $f\ (g\ x)\ y$
$h\ (\lambda x.\ f\ (g\ x))$

Convention: $f\ t_1\ t_2\ t_3 \ \equiv\ ((f\ t_1)\ t_2)\ t_3$

This language of terms is known as the *λ-calculus*.

The computation rule of the λ-calculus is the replacement of formal by actual parameters:

$$(\lambda x.\ t)\ u \ = \ t[u/x]$$

where $t[u/x]$ is "t with u substituted for x".

Example: $(\lambda x.\ x + \mathit{5})\ \mathit{3} \ = \ \mathit{3} + \mathit{5}$

- The step from $(\lambda x.\ t)\ u$ to $t[u/x]$ is called *β-reduction*.
- Isabelle performs β-reduction automatically.

Terms must be well-typed

(the argument of every function call must be of the right type)

Notation:

$t\ ::\ \tau$ means "t is a well-typed term of type τ".

$$\frac{t :: \tau_1 \Rightarrow \tau_2 \qquad u :: \tau_1}{t\ u :: \tau_2}$$

Type inference

Isabelle automatically computes the type of each variable in a term. This is called *type inference*.

In the presence of *overloaded* functions (functions with multiple types) this is not always possible.

User can help with *type annotations* inside the term.
Example: $f\ (x{::}nat)$

Currying

Thou shalt Curry your functions

- Curried: $f :: \tau_1 \Rightarrow \tau_2 \Rightarrow \tau$
- Tupled: $f' :: \tau_1 \times \tau_2 \Rightarrow \tau$

Advantage:

Currying allows *partial application*

$f\ a_1$ where $a_1 :: \tau_1$

Predefined syntactic sugar

- *Infix:* $+$, $-$, $*$, $\#$, $@$, ...
- *Mixfix:* $\mathit{if}\ \_\ \mathit{then}\ \_\ \mathit{else}\ \_$, $\mathit{case}\ \_\ \mathit{of}$, ...

Prefix binds more strongly than infix:

! $f\ x + y \ \equiv\ (f\ x) + y \ \not\equiv\ f\ (x + y)$ **!**

Enclose if and case in parentheses:

! $(\mathit{if}\ \_\ \mathit{then}\ \_\ \mathit{else}\ \_)$ **!**

Isabelle text = Theory = Module

Syntax:
```
theory MyTh
imports ImpTh_1 ... ImpTh_n
begin
(definitions, theorems, proofs, ...)*
end
```

$MyTh$: name of theory. Must live in file $MyTh$`.thy`

$ImpTh_i$: name of *imported* theories. Import transitive.

Usually: `imports Main`

Concrete syntax

In `.thy` files:

Types, terms and formulas need to be inclosed in "

Except for single identifiers

" normally not shown on slides

1 Overview of Isabelle/HOL

Types and terms

Interface

By example: types $bool$, nat and $list$

Summary

isabelle jedit

- Based on *jedit* editor
- Processes Isabelle text automatically when editing `.thy` files (like modern Java IDEs)

`Overview_Demo.thy`

1 Overview of Isabelle/HOL

Types and terms

Interface

By example: types $bool$, nat and $list$

Summary

Type $bool$

datatype $bool \; = \; True \; | \; False$

Predefined functions:
$\wedge, \vee, \longrightarrow, \ldots \; :: bool \Rightarrow bool \Rightarrow bool$

A *formula* is a term of type $bool$

if-and-only-if: $=$

Type nat

datatype $nat \;=\; 0 \;|\; Suc\; nat$

Values of type nat: 0, $Suc\ 0$, $Suc(Suc\ 0)$, $\ldots$

Predefined functions: $+$, $*$, ... $::\ nat \Rightarrow nat \Rightarrow nat$

! Numbers and arithmetic operations are overloaded:
$0,1,2,\ldots :: {'a}, \quad + :: {'a} \Rightarrow {'a} \Rightarrow {'a}$

You need type annotations: $1 :: nat$, $x + (y{::}nat)$
unless the context is unambiguous: $Suc\ z$

Nat_Demo.thy

An informal proof

Lemma $add\ m\ 0 = m$
Proof by induction on m.

- Case 0 (the base case):
  $add\ 0\ 0 = 0$ holds by definition of add.
- Case $Suc\ m$ (the induction step):
  We assume $add\ m\ 0 = m$,
  the induction hypothesis (IH).
  We need to show $add\ (Suc\ m)\ 0 = Suc\ m$.
  The proof is as follows:
  $$\begin{array}{lcll} add\ (Suc\ m)\ 0 & = & Suc\ (add\ m\ 0) & \text{by def. of } add \\ & = & Suc\ m & \text{by IH} \end{array}$$

Type $'a\ list$

Lists of elements of type $'a$

datatype $'a\ list\ =\ Nil\ |\ Cons\ 'a\ ('a\ list)$

Some lists: Nil, $Cons\ 1\ Nil$, $Cons\ 1\ (Cons\ 2\ Nil)$, ...

Syntactic sugar:

- $[] = Nil$: empty list
- $x\ \#\ xs = Cons\ x\ xs$:
  list with first element x (*"head"*) and rest xs (*"tail"*)
- $[x_1, \ldots, x_n] = x_1\ \#\ \ldots\ x_n\ \#\ []$

Structural Induction for lists

To prove that $P(xs)$ for all lists xs, prove

- $P([])$ and
- for arbitrary x and xs, $P(xs)$ implies $P(x\#xs)$.

$$\frac{P([]) \qquad \bigwedge x\ xs.\ P(xs) \Longrightarrow P(x\#xs)}{P(xs)}$$

List_Demo.thy

An informal proof

Lemma $app\ (app\ xs\ ys)\ zs = app\ xs\ (app\ ys\ zs)$

Proof by induction on xs.

- Case Nil: $app\ (app\ Nil\ ys)\ zs = app\ ys\ zs = app\ Nil\ (app\ ys\ zs)$ holds by definition of app.
- Case $Cons\ x\ xs$: We assume $app\ (app\ xs\ ys)\ zs = app\ xs\ (app\ ys\ zs)$ (IH), and we need to show
  $app\ (app\ (Cons\ x\ xs)\ ys)\ zs =$
  $app\ (Cons\ x\ xs)\ (app\ ys\ zs)$.
  The proof is as follows:
  $app\ (app\ (Cons\ x\ xs)\ ys)\ zs$
  $= Cons\ x\ (app\ (app\ xs\ ys)\ zs)$ by definition of app
  $= Cons\ x\ (app\ xs\ (app\ ys\ zs))$ by IH
  $= app\ (Cons\ x\ xs)\ (app\ ys\ zs)$ by definition of app

Large library: `HOL/List.thy`

Included in `Main`.

Don't reinvent, reuse!

Predefined: $xs\ @\ ys$ (append), $length$, and map:

$$map\ f\ [x_1, \ldots, x_n] = [f\ x_1, \ldots, f\ x_n]$$

fun $map :: ('a \Rightarrow 'b) \Rightarrow 'a\ list \Rightarrow 'b\ list$ **where**
$map\ f\ [] = []\ \ |$
$map\ f\ (x \# xs) = f\ x\ \#\ map\ f\ xs$

Note: map takes *function* as argument.

- **datatype** defines (possibly) recursive data types.
- **fun** defines (possibly) recursive functions by pattern-matching over datatype constructors.

Proof methods

- $induction$ performs structural induction on some variable (if the type of the variable is a datatype).
- $auto$ solves as many subgoals as it can, mainly by simplification (symbolic evaluation):

  "$=$" is used only from left to right!

Proofs

General schema:

lemma $name$: `"..."`
apply `(...)`
apply `(...)`
⋮
done

If the lemma is suitable as a simplification rule:

lemma $name$`[simp]:  "..."`

Top down proofs

Command

sorry

"completes" any proof.

Allows top down development:

Assume lemma first, prove it later.

The proof state

$$1.\ \bigwedge x_1 \ \dots \ x_p.\ \ A \Longrightarrow B$$

$x_1 \ \dots \ x_p$	fixed local variables
A	local assumption(s)
B	actual (sub)goal

Preview: Multiple assumptions

$$[\![A_1; \dots ; A_n]\!] \Longrightarrow B$$

abbreviates

$$A_1 \Longrightarrow \dots \Longrightarrow A_n \Longrightarrow B$$

$;\quad \approx \quad$ "and"

1. Overview of Isabelle/HOL
2. Type and function definitions
3. Induction Heuristics
4. Simplification

2 Type and function definitions

Type definitions

Function definitions

Type synonyms

type_synonym $name = \tau$

Introduces a *synonym* $name$ for type τ

Examples:

type_synonym $string = char\ list$

type_synonym $('a,'b)foo = 'a\ list \times\ 'b\ list$

Type synonyms are expanded after parsing
and are not present in internal representation and output

datatype — the general case

$$\textbf{datatype}\ (\alpha_1,\ldots,\alpha_n)\tau \;=\; C_1\ \tau_{1,1}\ldots\tau_{1,n_1} \;\mid\; \ldots \;\mid\; C_k\ \tau_{k,1}\ldots\tau_{k,n_k}$$

- *Types:* $C_i :: \tau_{i,1} \Rightarrow \cdots \Rightarrow \tau_{i,n_i} \Rightarrow (\alpha_1,\ldots,\alpha_n)\tau$
- *Distinctness:* $C_i\ \ldots \neq C_j\ \ldots$ if $i \neq j$
- *Injectivity:* $(C_i\ x_1\ldots x_{n_i} = C_i\ y_1\ldots y_{n_i}) = (x_1 = y_1 \wedge \cdots \wedge x_{n_i} = y_{n_i})$

Distinctness and injectivity are applied automatically
Induction must be applied explicitly

Case expressions

Datatype values can be taken apart with $case$:

$$(case\ xs\ of\ \ [] \Rightarrow \ldots\ \ |\ \ y\#ys \Rightarrow\ ...\ y\ ...\ ys\ ...)$$

Wildcards: _

$$(case\ m\ of\ \ 0 \Rightarrow Suc\ 0\ \ |\ \ Suc\ \_ \Rightarrow 0)$$

Nested patterns:

$$(case\ xs\ of\ \ [0] \Rightarrow 0\ \ |\ \ [Suc\ n] \Rightarrow n\ \ |\ \ \_ \Rightarrow 2)$$

Complicated patterns mean complicated proofs!

Need ( ) in context

Tree_Demo.thy

The *option* type

datatype $'a\ option = None \mid Some\ 'a$

If $'a$ has values $a_1,\ a_2,\ \ldots$
then $'a\ option$ has values $None$, $Some\ a_1$, $Some\ a_2$, $\ldots$

Typical application:

fun $lookup :: ('a \times\ 'b)\ list \Rightarrow\ 'a \Rightarrow\ 'b\ option$ **where**
$lookup\ []\ x = None\ |$
$lookup\ ((a,b)\ \#\ ps)\ x =$
$\quad (if\ a = x\ then\ Some\ b\ else\ lookup\ ps\ x)$

2 Type and function definitions

Type definitions

Function definitions

Non-recursive definitions

Example:

definition $sq :: nat \Rightarrow nat$ **where** $sq\ n\ =\ n{*}n$

No pattern matching, just $f\,x_1\,\ldots\,x_n\ =\ \ldots$

The danger of nontermination

How about $f\, x = f\, x + 1$?

! All functions in HOL must be total **!**

Key features of **fun**

- Pattern-matching over datatype constructors
- Order of equations matters
- Termination must be provable automatically by size measures
- Proves customized induction schema

Example: separation

fun $sep :: 'a \Rightarrow 'a\ list \Rightarrow 'a\ list$ **where**
$sep\ a\ (x\#y\#zs) = x\ \#\ a\ \#\ sep\ a\ (y\#zs)\ \ |$
$sep\ a\ xs = xs$

Example: Ackermann

fun $ack :: nat \Rightarrow nat \Rightarrow nat$ **where**

$$
\begin{array}{llll}
ack\ 0 & n & = Suc\ n & | \\
ack\ (Suc\ m) & 0 & = ack\ m\ (Suc\ 0) & | \\
ack\ (Suc\ m) & (Suc\ n) & = ack\ m\ (ack\ (Suc\ m)\ n) &
\end{array}
$$

Terminates because the arguments decrease *lexicographically* with each recursive call:

- $(Suc\ m,\ 0) > (m,\ Suc\ 0)$
- $(Suc\ m,\ Suc\ n) > (Suc\ m,\ n)$
- $(Suc\ m,\ Suc\ n) > (m,\ \_)$

primrec

- A restrictive version of **fun**
- Means *primitive recursive*
- Most functions are primitive recursive
- Frequently found in Isabelle theories

The essence of primitive recursion:

$$
\begin{array}{lll}
f(0) & = \dots & \text{no recursion} \\
f(Suc\ n) & = \dots f(n) \dots &
\end{array}
$$

$$
\begin{array}{lll}
g([]) & = \dots & \text{no recursion} \\
g(x\#xs) & = \dots g(xs) \dots &
\end{array}
$$

1. Overview of Isabelle/HOL
2. Type and function definitions
3. **Induction Heuristics**
4. Simplification

Basic induction heuristics

Theorems about recursive functions are proved by induction

Induction on argument number i of f
if f is defined by recursion on argument number i

A tail recursive reverse

Our initial reverse:

fun $rev :: {'a}\ list \Rightarrow {'a}\ list$ **where**
$rev\ []\ = []\ \mid$
$rev\ (x \# xs)\ = rev\ xs\ @\ [x]$

A tail recursive version:

fun $itrev :: {'a}\ list \Rightarrow {'a}\ list \Rightarrow {'a}\ list$ **where**
$itrev\ []\ \ ys = ys\ \mid$
$itrev\ (x \# xs)\ \ ys =$

lemma $itrev\ xs\ [] = rev\ xs$

`Induction_Demo.thy`

Generalisation

Generalisation

- Replace constants by variables
- Generalize free variables
  - by $arbitrary$ in induction proof
  - (or by universal quantifier in formula)

So far, all proofs were by structural induction
because all functions were primitive recursive.

In each induction step, 1 constructor is added.
In each recursive call, 1 constructor is removed.

Now: induction for complex recursion patterns.

Computation Induction: Example

fun $div2 :: nat \Rightarrow nat$ **where**
$div2\ 0 = 0\ \ |$
$div2\ (Suc\ 0) = 0\ \ |$
$div2\ (Suc(Suc\ n)) = Suc(div2\ n)$

$\leadsto$ induction rule `div2.induct`:

$$\frac{P(0) \quad P(Suc\ 0) \quad \bigwedge n.\ \ P(n) \Longrightarrow P(Suc(Suc\ n))}{P(m)}$$

Computation Induction

If $f :: \tau \Rightarrow \tau'$ is defined by **fun**, a special induction schema is provided to prove $P(x)$ for all $x :: \tau$:

for each defining equation

$$f(e) \;=\; \ldots f(r_1) \ldots f(r_k) \ldots$$

prove $P(e)$ *assuming* $P(r_1)$, $\ldots$, $P(r_k)$.

Induction follows course of (terminating!) computation
Motto: properties of f are best proved by rule $f.induct$

How to apply $f.induct$

If $f :: \tau_1 \Rightarrow \cdots \Rightarrow \tau_n \Rightarrow \tau'$:

$$(induction\ a_1\ \ldots\ a_n\ rule{:}\ f.induct)$$

Heuristic:

- there should be a call $f\ a_1\ \ldots\ a_n$ in your goal
- ideally the a_i should be variables.

`Induction_Demo.thy`

Computation Induction

1. Overview of Isabelle/HOL
2. Type and function definitions
3. Induction Heuristics
4. Simplification

Simplification means . . .

Using equations $l = r$ from left to right

As long as possible

Terminology: equation $\leadsto$ *simplification rule*

Simplification = (Term) Rewriting

An example

Equations:

$$
\begin{aligned}
0 + n &= n & (1)\\
(Suc\ m) + n &= Suc\ (m + n) & (2)\\
(Suc\ m \leq Suc\ n) &= (m \leq n) & (3)\\
(0 \leq m) &= True & (4)
\end{aligned}
$$

Rewriting:

$$
\begin{aligned}
0 + Suc\ 0 &\leq Suc\ 0 + x & \stackrel{(1)}{=}\\
Suc\ 0 &\leq Suc\ 0 + x & \stackrel{(2)}{=}\\
Suc\ 0 &\leq Suc\ (0 + x) & \stackrel{(3)}{=}\\
0 &\leq 0 + x & \stackrel{(4)}{=}\\
&True &
\end{aligned}
$$

Conditional rewriting

Simplification rules can be conditional:

$$\llbracket\ P_1;\ \ldots;\ P_k\ \rrbracket \Longrightarrow l = r$$

is applicable only if all P_i can be proved first, again by simplification.

Example:

$$\begin{array}{rrcl} & p(0) & = & True \\ p(x) \Longrightarrow & f(x) & = & g(x) \end{array}$$

We can simplify $f(0)$ to $g(0)$ but we cannot simplify $f(1)$ because $p(1)$ is not provable.

Termination

Simplification may not terminate.
Isabelle uses $simp$-rules (almost) blindly from left to right.

Example: $f(x) = g(x),\ g(x) = f(x)$

$$[\![\ P_1; \ldots; P_k\ ]\!] \Longrightarrow l = r$$

is suitable as a $simp$-rule only
if l is "bigger" than r and each P_i

$$n < m \Longrightarrow (n < Suc\ m) = True \quad \text{YES}$$
$$Suc\ n < m \Longrightarrow (n < m) = True \quad \text{NO}$$

Proof method $simp$

Goal: 1. $[\![P_1; \ldots; P_m]\!] \Longrightarrow C$

apply$(simp\ add{:}\ eq_1\ \ldots\ eq_n)$

Simplify $P_1\ \ldots\ P_m$ and C using

- lemmas with attribute $simp$
- rules from **fun** and **datatype**
- additional lemmas $eq_1\ \ldots\ eq_n$
- assumptions $P_1\ \ldots\ P_m$

Variations:

- $(simp\ \ldots\ del{:}\ \ldots)$ removes $simp$-lemmas
- add and del are optional

$auto$ versus $simp$

- $auto$ acts on all subgoals
- $simp$ acts only on subgoal 1

- $auto$ applies $simp$ and more

- $auto$ can also be modified:
  $(auto\ simp\ add\!:\ \ldots\ \ simp\ del\!:\ \ldots)$

Rewriting with definitions

Definitions (**definition**) must be used explicitly:

$$(simp\ add{:}\ f\_def \ldots)$$

f is the function whose definition is to be unfolded.

Case splitting with $simp$

Automatic:

$$P(\mathit{if}\ A\ \mathit{then}\ s\ \mathit{else}\ t)$$
$$=$$
$$(A \longrightarrow P(s)) \wedge (\neg A \longrightarrow P(t))$$

By hand:

$$P(\mathit{case}\ e\ \mathit{of}\ \mathit{0} \Rightarrow a \mid \mathit{Suc}\ n \Rightarrow b)$$
$$=$$
$$(e = \mathit{0} \longrightarrow P(a)) \wedge (\forall\, n.\ e = \mathit{Suc}\ n \longrightarrow P(b))$$

Proof method: $(\mathit{simp\ split}\colon \mathit{nat.split})$

Or $auto$. Similar for any datatype t: $\mathit{t.split}$

`Simp_Demo.thy`

Chapter 3

Case Study: IMP Expressions

5 Case Study: IMP Expressions

5 Case Study: IMP Expressions

This section introduces

arithmetic and boolean expressions

of our imperative language IMP.

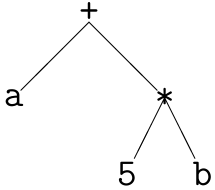
IMP *commands* are introduced later.

5. Case Study: IMP Expressions
   - Arithmetic Expressions
   - Boolean Expressions
   - Stack Machine and Compilation

Concrete and abstract syntax

Concrete syntax: strings, eg `"a+5*b"`

Abstract syntax: trees, eg



Parser: function from strings to trees

Linear view of trees: terms, eg $Plus\ a\ (Times\ 5\ b)$

Abstract syntax trees/terms are datatype values!

Concrete syntax is defined by a context-free grammar, eg

$$a ::= n \mid x \mid (a) \mid a + a \mid a * a \mid \ldots$$

where n can be any natural number and x any variable.

We focus on *abstract* syntax
which we introduce via datatypes.

Datatype $aexp$

Variable names are strings, values are integers:

type_synonym $vname = string$
datatype $aexp = N\ int \mid V\ vname \mid Plus\ aexp\ aexp$

Concrete	Abstract
`5`	$N\ 5$
`x`	$V\ ''x''$
`x+y`	$Plus\ (V\ ''x'')\ (V\ ''y'')$
`2+(z+3)`	$Plus\ (N\ 2)\ (Plus\ (V\ ''z'')\ (N\ 3))$

Warning

This is syntax, not (yet) semantics!

$$N\ \mathit{0} \neq \mathit{Plus}\ (N\ \mathit{0})\ (N\ \mathit{0})$$

The (program) state

What is the value of `x+1`?

- The value of an expression depends on the value of its variables.
- The value of all variables is recorded in the *state*.
- The state is a function from variable names to values:

type_synonym $val = int$
type_synonym $state = vname \Rightarrow val$

Function update notation

If $f :: \tau_1 \Rightarrow \tau_2$ and $a :: \tau_1$ and $b :: \tau_2$ then

$$f(a := b)$$

is the function that behaves like f
except that it returns b for argument a.

$$f(a := b) = (\lambda x.\ \mathit{if}\ x = a\ \mathit{then}\ b\ \mathit{else}\ f\ x)$$

How to write down a state

Some states:

- $\lambda x.\ 0$
- $(\lambda x.\ 0)(''a'' := 3)$
- $((\lambda x.\ 0)(''a'' := 5))(''x'' := 3)$

Nicer notation:

$$<''a'' := 5,\ ''x'' := 3,\ ''y'' := 7>$$

Maps everything to 0, but $''a''$ to 5, $''x''$ to 3, etc.

AExp.thy

5 Case Study: IMP Expressions

- Arithmetic Expressions
- Boolean Expressions
- Stack Machine and Compilation

BExp.thy

5 Case Study: IMP Expressions

Arithmetic Expressions

Boolean Expressions

Stack Machine and Compilation

ASM.thy

This was easy.
Because evaluation of expressions always terminates.
But execution of programs may *not* terminate.
Hence we cannot define it by a total recursive function.

We need more logical machinery
to define program execution and reason about it.

Chapter 4

Logic and Proof Beyond Equality

6. Logical Formulas

7. Proof Automation

8. Single Step Proofs

9. Inductive Definitions

6 Logical Formulas

7 Proof Automation

8 Single Step Proofs

9 Inductive Definitions

Syntax (in decreasing precedence):

$$
\begin{array}{lcllll}
\mathit{form} & ::= & (\mathit{form}) & \mid\ \mathit{term} = \mathit{term} & \mid & \neg\mathit{form} \\
& \mid & \mathit{form} \wedge \mathit{form} & \mid\ \mathit{form} \vee \mathit{form} & \mid & \mathit{form} \longrightarrow \mathit{form} \\
& \mid & \forall x.\ \mathit{form} & \mid\ \exists x.\ \mathit{form} & &
\end{array}
$$

Examples:

$$
\begin{array}{rcl}
\neg\ A\ \wedge\ B \vee\ C & \equiv & ((\neg\ A)\ \wedge\ B)\ \vee\ C \\
s\ =\ t\ \wedge\ C & \equiv & (s\ =\ t)\ \wedge\ C \\
A\ \wedge\ B = B\ \wedge\ A & \equiv & A\ \wedge\ (B =\ B)\ \wedge\ A \\
\forall\, x.\ P\ x \wedge\ Q\ x & \equiv & \forall\, x.\ (P\ x \wedge\ Q\ x)
\end{array}
$$

Input syntax: $\longleftrightarrow$ (same precedence as $\longrightarrow$)

Variable binding convention:

$$\forall\, x\ y.\ P\ x\ y \ \equiv\ \forall\, x.\ \forall\, y.\ P\ x\ y$$

Similarly for $\exists$ and λ.

Warning

Quantifiers have low precedence
and need to be parenthesized (if in some context)

$$! \quad P \wedge \forall x.\ Q\ x \ \leadsto\ P \wedge (\forall x.\ Q\ x) \quad !$$

Mathematical symbols

. . . and their ascii representations:

$\forall$	\<forall>	ALL
$\exists$	\<exists>	EX
λ	\<lambda>	%
$\longrightarrow$	-->	
$\longleftrightarrow$	<-->	
$\wedge$	/\	&
$\vee$	\/	|
$\neg$	\<not>	~
$\neq$	\<noteq>	~=

Sets over type $'a$

$'a\ set$

- $\{\}, \quad \{e_1,\ldots,e_n\}$
- $e \in A, \quad A \subseteq B$
- $A \cup B, \quad A \cap B, \quad A - B, \quad -\ A$
- $\ldots$

$\in$	`\<in>`	`:`
$\subseteq$	`\<subseteq>`	`<=`
$\cup$	`\<union>`	`Un`
$\cap$	`\<inter>`	`Int`

Set comprehension

- $\{x.\ P\}$ where x is a variable
- But not $\{t.\ P\}$ where t is a proper term
- Instead: $\{t \mid x\ y\ z.\ P\}$
  is short for $\{v.\ \exists\, x\ y\ z.\ v = t \wedge P\}$
  where x, y, z are the free variables in t

simp and *auto*

simp: rewriting and a bit of arithmetic

auto: rewriting and a bit of arithmetic, logic and sets

- Show you where they got stuck
- highly incomplete
- Extensible with new *simp*-rules

Exception: *auto* acts on all subgoals

fastforce

- rewriting, logic, sets, relations and a bit of arithmetic.
- incomplete but better than *auto*.
- Succeeds or fails
- Extensible with new *simp*-rules

blast

- A complete proof search procedure for FOL ...
- ... but (almost) without "$=$"
- Covers logic, sets and relations
- Succeeds or fails
- Extensible with new deduction rules

Automating arithmetic

$arith$:

- proves linear formulas (no "$*$")
- complete for quantifier-free $real$ arithmetic
- complete for first-order theory of nat and int (Presburger arithmetic)

Sledgehammer

Architecture:

Isabelle

Goal & filtered library $\downarrow$ $\uparrow$ Proof

external **ATPs**[1]

Characteristics:

- Sometimes it works,
- sometimes it doesn't.

Do you feel lucky?

[1] Automatic Theorem Provers

by$(proof\text{-}method)$

$\approx$

apply$(proof\text{-}method)$
done

Auto_Proof_Demo.thy

Step-by-step proofs can be necessary if automation fails and you have to explore where and why it failed by taking the goal apart.

What are these *?-variables* ?

After you have finished a proof, Isabelle turns all free variables V in the theorem into $?V$.

Example: theorem `conjI`: $[\![?P;\ ?Q]\!] \Longrightarrow ?P \wedge ?Q$

These ?-variables can later be instantiated:

- By hand:
  `conjI[of "a=b" "False"]` $\leadsto$
  $[\![a = b;\ False]\!] \Longrightarrow a = b \wedge False$
- By unification:
  unifying $?P \wedge ?Q$ with $a{=}b \wedge False$
  sets $?P$ to $a{=}b$ and $?Q$ to $False$.

Rule application

Example: rule: $[\![?P;\ ?Q]\!] \Longrightarrow ?P \wedge ?Q$

subgoal: $1.\ \ldots \Longrightarrow A \wedge B$

Result: $1.\ \ldots \Longrightarrow A$

$2.\ \ldots \Longrightarrow B$

The general case: applying rule $[\![\ A_1;\ \ldots\ ;\ A_n\ ]\!] \Longrightarrow A$ to subgoal $\ldots \Longrightarrow C$:

- Unify A and C
- Replace C with n new subgoals $A_1 \ldots A_n$

apply$(rule\ xyz)$

"Backchaining"

Typical backwards rules

$$\frac{?P \quad ?Q}{?P \wedge\ ?Q}\ \texttt{conjI}$$

$$\frac{?P \Longrightarrow\ ?Q}{?P \longrightarrow\ ?Q}\ \texttt{impI} \qquad \frac{\bigwedge x.\ ?P\ x}{\forall\, x.\ ?P\ x}\ \texttt{allI}$$

$$\frac{?P \Longrightarrow\ ?Q \quad ?Q \Longrightarrow\ ?P}{?P =\ ?Q}\ \texttt{iffI}$$

They are known as introduction rules
because they *introduce* a particular connective.

Teaching *blast* new intro rules

If r is a theorem $\llbracket A_1; \ldots; A_n \rrbracket \Longrightarrow A$ then

$$(\mathit{blast\ intro:\ r})$$

allows *blast* to backchain on r during proof search.

Example:

theorem trans: $\llbracket\ ?x \le\ ?y;\ ?y \le\ ?z\ \rrbracket \Longrightarrow\ ?x \le\ ?z$

goal $\mathit{1.}\ \llbracket\ a \le b;\ b \le c;\ c \le d\ \rrbracket \Longrightarrow a \le d$

proof **apply**$(\mathit{blast\ intro:\ trans})$

Can greatly increase the search space!

Forward proof: OF

If r is a theorem $A \Longrightarrow B$
and s is a theorem that unifies with A then

$$r[OF\ s]$$

is the theorem obtained by proving A with s.

Example: theorem `refl`: $?t = ?t$

`conjI[OF refl[of "a"]]`

$$\leadsto$$

$$?Q \Longrightarrow a = a \wedge ?Q$$

The general case:

If r is a theorem $[\![\ A_1; \ldots; A_n\ ]\!] \Longrightarrow A$
and $r_1, \ldots, r_m$ $(m{\le}n)$ are theorems then

$$r[OF\ r_1\ \ldots\ r_m]$$

is the theorem obtained
by proving $A_1\ \ldots\ A_m$ with $r_1\ \ldots\ r_m$.

Example: theorem `refl`: $?t = ?t$

```
conjI[OF refl[of "a"] refl[of "b"]]
```

$$\leadsto$$

$$a = a \wedge b = b$$

From now on: $?$ mostly suppressed on slides

Single_Step_Demo.thy

$\Longrightarrow$ versus $\longrightarrow$

$\Longrightarrow$ is part of the Isabelle framework. It structures theorems and proof states: $[\![A_1; \ldots; A_n]\!] \Longrightarrow A$

$\longrightarrow$ is part of HOL and can occur inside the logical formulas A_i and A.

Phrase theorems like this $[\![A_1; \ldots; A_n]\!] \Longrightarrow A$
not like this $A_1 \wedge \ldots \wedge A_n \longrightarrow A$

Example: even numbers

Informally:

- 0 is even
- If n is even, so is $n + 2$
- These are the only even numbers

In Isabelle/HOL:

inductive $ev :: nat \Rightarrow bool$
where
$ev\ 0\ \ |$
$ev\ n \Longrightarrow ev\ (n + 2)$

An easy proof: $ev\ 4$

$$ev\ 0 \Longrightarrow ev\ 2 \Longrightarrow ev\ 4$$

Consider

fun $\mathit{even} :: \mathit{nat} \Rightarrow \mathit{bool}$ **where**
$\mathit{even}\ 0 = \mathit{True} \mid$
$\mathit{even}\ (\mathit{Suc}\ 0) = \mathit{False} \mid$
$\mathit{even}\ (\mathit{Suc}\ (\mathit{Suc}\ n)) = \mathit{even}\ n$

A trickier proof: $\mathit{ev}\ m \Longrightarrow \mathit{even}\ m$

By induction on the *structure* of the derivation of $\mathit{ev}\ m$

Two cases: $\mathit{ev}\ m$ is proved by

- rule $\mathit{ev}\ 0$
  $\Longrightarrow m = 0 \Longrightarrow \mathit{even}\ m = \mathit{True}$
- rule $\mathit{ev}\ n \Longrightarrow \mathit{ev}\ (n{+}2)$
  $\Longrightarrow m = n{+}2$ and $\mathit{even}\ n$ (IH)
  $\Longrightarrow \mathit{even}\ m = \mathit{even}\ (n{+}2) = \mathit{even}\ n = \mathit{True}$

Rule induction for ev

To prove

$$ev\ n \Longrightarrow P\ n$$

by *rule induction* on $ev\ n$ we must prove

- $P\ 0$
- $P\ n \Longrightarrow P(n+2)$

Rule `ev.induct`:

$$\frac{ev\ n \quad P\ 0 \quad \bigwedge n.\ [\![\ ev\ n;\ P\ n\ ]\!] \Longrightarrow P(n+2)}{P\ n}$$

Format of inductive definitions

inductive $I :: \tau \Rightarrow bool$ **where**

$\llbracket\ I\ a_1; \ \ldots\ ; I\ a_n\ \rrbracket \Longrightarrow I\ a \quad |$

$\vdots$

Note:

- I may have multiple arguments.
- Each rule may also contain *side conditions* not involving I.

Rule induction in general

To prove

$$I\ x \Longrightarrow P\ x$$

by *rule induction* on $I\ x$
we must prove for every rule

$$[\![\ I\ a_1; \ \ldots\ ; I\ a_n\ ]\!] \Longrightarrow I\ a$$

that P is preserved:

$$[\![\ I\ a_1;\ P\ a_1; \ \ldots\ ; I\ a_n;\ P\ a_n\ ]\!] \Longrightarrow P\ a$$

! Rule induction is absolutely central
to (operational) semantics
and the rest of this lecture course **!**

Inductive_Demo.thy

Inductively defined sets

inductive_set $I :: \tau\ set$ **where**

$\llbracket\ a_1 \in I; \ \ldots\ ; a_n \in I\ \rrbracket \Longrightarrow a \in I \ \mid$

$\vdots$

Difference to **inductive**:

- arguments of I are tupled, not curried
- I can later be used with set theoretic operators, eg $I \cup \ldots$

Chapter 5

Isar: A Language for Structured Proofs

10. Isar by example

11. Proof patterns

12. Streamlining Proofs

13. Proof by Cases and Induction

Apply scripts

- unreadable
- hard to maintain
- do not scale

No structure!

Apply scripts versus Isar proofs

Apply script = assembly language program

Isar proof = structured program with comments

But: **apply** still useful for proof exploration

A typical Isar proof

proof
  assume $formula_0$
  have $formula_1$ **by** $simp$
  $\vdots$
  have $formula_n$ **by** $blast$
  show $formula_{n+1}$ **by** $\ldots$
qed

proves $formula_0 \Longrightarrow formula_{n+1}$

Isar core syntax

$$
\begin{array}{lcl}
\text{proof} & = & \textbf{proof}\ [\text{method}]\ \ \text{step}^*\ \ \textbf{qed} \\
 & | & \textbf{by}\ \text{method}
\end{array}
$$

$$
\text{method} \;=\; (\mathit{simp}\ \ldots) \mid (\mathit{blast}\ \ldots) \mid (\mathit{induction}\ \ldots) \mid \ldots
$$

$$
\begin{array}{lcll}
\text{step} & = & \textbf{fix}\ \text{variables} & (\bigwedge) \\
 & | & \textbf{assume}\ \text{prop} & (\Longrightarrow) \\
 & | & [\textbf{from}\ \text{fact}^+]\ \ (\textbf{have} \mid \textbf{show})\ \text{prop}\ \ \text{proof} &
\end{array}
$$

$$
\text{prop} \;=\; [\text{name:}]\ \text{"formula"}
$$

$$
\text{fact} \;=\; \text{name} \mid \ldots
$$

Example: Cantor's theorem

lemma $\neg\ surj(f :: 'a \Rightarrow 'a\ set)$
proof default proof: assume *surj*, show *False*
  assume a: $surj\ f$
  from a **have** b: $\forall\ A.\ \exists\ a.\ A = f\ a$
    by$(simp\ add\colon surj\_def)$
  from b **have** c: $\exists\ a.\ \{x.\ x \notin f\ x\} = f\ a$
    by $blast$
  from c **show** $False$
    by $blast$
qed

Isar_Demo.thy

Cantor and abbreviations

Abbreviations

$this$	$=$	the previous proposition proved or assumed
then	$=$	**from** $this$
thus	$=$	**then show**
hence	$=$	**then have**

using and **with**

(**have**|**show**) prop **using** facts

=

from facts (**have**|**show**) prop

with facts

=

from facts *this*

Structured lemma statement

lemma
  fixes $f :: 'a \Rightarrow 'a\ set$
  assumes s: $surj\ f$
  shows $False$
proof $-$ no automatic proof step
  have $\exists\ a.\ \{x.\ x \notin f\ x\} = f\ a$ **using** s
    by$(auto\ simp\text{:}\ surj\_def)$
  thus $False$ **by** $blast$
qed

Proves $surj\ f \Longrightarrow False$
but $surj\ f$ *becomes local fact* s *in proof.*

The essence of structured proofs

Assumptions and intermediate facts
can be named and referred to explicitly and selectively

Structured lemma statements

fixes $x :: \tau_1$ **and** $y :: \tau_2 \ldots$
assumes *a:* P **and** *b:* $Q \ldots$
shows R

- **fixes** and **assumes** sections optional
- **shows** optional if no **fixes** and **assumes**

10 Isar by example

11 Proof patterns

12 Streamlining Proofs

13 Proof by Cases and Induction

Case distinction

show R
proof $cases$
  assume P
  $\vdots$
  show R $\ldots$
next
  assume $\neg$ P
  $\vdots$
  show R $\ldots$
qed

have $P \vee Q$ $\ldots$
then show R
proof
  assume P
  $\vdots$
  show R $\ldots$
next
  assume Q
  $\vdots$
  show R $\ldots$
qed

Contradiction

show $\neg\ P$
proof
  assume P
  $\vdots$
  show $False\ \ \ldots$
qed

show P
proof $(rule\ ccontr)$
  assume $\neg P$
  $\vdots$
  show $False\ \ \ldots$
qed

$\longleftrightarrow$

show $P \longleftrightarrow Q$
proof
  assume P
  $\vdots$
  show Q $\ldots$
next
  assume Q
  $\vdots$
  show P $\ldots$
qed

$\forall$ and $\exists$ introduction

show $\forall\, x.\ P(x)$
proof
  fix x local fixed variable
  show $P(x)$ $\ldots$
qed

show $\exists\, x.\ P(x)$
proof
  $\vdots$
  show $P(witness)$ $\ldots$
qed

$\exists$ elimination: **obtain**

have $\exists\, x.\ P(x)$
then obtain x **where** *p:* $P(x)$ **by** *blast*
$\vdots$ $\quad x$ fixed local variable

Works for one or more x

obtain example

lemma $\neg\ surj(f :: 'a \Rightarrow 'a\ set)$
proof
  assume $surj\ f$
  hence $\exists\, a.\ \{x.\ x \notin f\ x\} = f\ a$ **by**$(auto\ simp\!: surj\_def)$
  then obtain a **where** $\{x.\ x \notin f\ x\} = f\ a$ **by** $blast$
  hence $a \notin f\ a \longleftrightarrow a \in f\ a$ **by** $blast$
  thus $False$ **by** $blast$
qed

Set equality and subset

show $A = B$
proof
  show $A \subseteq B$ $\ldots$
next
  show $B \subseteq A$ $\ldots$
qed

show $A \subseteq B$
proof
  fix x
  assume $x \in A$
  $\vdots$
  show $x \in B$ $\ldots$
qed

`Isar_Demo.thy`

Exercise

12 Streamlining Proofs

Pattern Matching and Quotations

Top down proof development

moreover and raw proof blocks

Example: pattern matching

show $formula_1 \longleftrightarrow formula_2$ (**is** $?L \longleftrightarrow ?R$)
proof
  assume $?L$
  $\vdots$
  show $?R$ ...
next
  assume $?R$
  $\vdots$
  show $?L$ ...
qed

$?thesis$

show *formula* (is $?thesis$)
proof -
$\vdots$
show $?thesis$ ...
qed

Every show implicitly defines $?thesis$

let

Introducing local abbreviations in proofs:

let $?t$ = *"some-big-term"*

$\vdots$

have *"*$\ldots$ $?t$ $\ldots$*"*

Quoting facts by value

By name:

have *x0:* "$x > 0$" ...

⋮

from *x0* ...

By value:

have "$x > 0$" ...

⋮

from \`$x{>}0$\` ...

↑ ↑

back quotes

Isar_Demo.thy

Pattern matching and quotation

12 Streamlining Proofs

Pattern Matching and Quotations

Top down proof development

moreover and raw proof blocks

Example

lemma

$$(\exists\, ys\ zs.\ xs = ys\ @\ zs \wedge \mathit{length}\ ys = \mathit{length}\ zs) \vee$$
$$(\exists\, ys\ zs.\ xs = ys\ @\ zs \wedge \mathit{length}\ ys = \mathit{length}\ zs + \mathit{1})$$

proof ???

Isar_Demo.thy

Top down proof development

When automation fails

Split proof up into smaller steps.

Or explore by **apply**:

have . . . **using** . . .	
apply -	to make incoming facts part of proof state
apply *auto*	or whatever
apply . . .	

At the end:

- **done**
- Better: convert to structured proof

12 Streamlining Proofs

Pattern Matching and Quotations

Top down proof development

moreover and raw proof blocks

moreover—ultimately

have P_1 $\ldots$
moreover
have P_2 $\ldots$
moreover
$\vdots$
moreover
have P_n $\ldots$
ultimately
have P $\ldots$

$\approx$

have lab_1: P_1 $\ldots$
have lab_2: P_2 $\ldots$
$\vdots$
have lab_n: P_n $\ldots$
from lab_1 lab_2 $\ldots$
have P $\ldots$

With names

Raw proof blocks

$\{$ **fix** $x_1 \ \dots \ x_n$
assume $A_1 \ \dots \ A_m$
$\vdots$
have B
$\}$

proves $[\![\ A_1; \ \dots \ ; \ A_m \ ]\!] \Longrightarrow B$
where all x_i have been replaced by $?x_i$.

Isar_Demo.thy

moreover and $\{\quad\}$

Proof state and Isar text

In general: **proof** *method*

Applies *method* and generates subgoal(s):

$$\bigwedge x_1 \ \ldots \ x_n \ [\![\ A_1; \ \ldots \ ; \ A_m \ ]\!] \Longrightarrow B$$

How to prove each subgoal:

fix $x_1 \ \ldots \ x_n$
assume $A_1 \ \ldots \ A_m$
$\vdots$
show B

Separated by **next**

`Isar_Induction_Demo.thy`

Proof by cases

Datatype case analysis

datatype $t \;=\; C_1 \; \vec{\tau} \;\mid\; \ldots$

proof $(cases$ "$term$"$)$
  case $(C_1 \; x_1 \; \ldots \; x_k)$
  $\ldots \; x_j \; \ldots$
next
$\vdots$
qed

where **case** $(C_i \; x_1 \; \ldots \; x_k) \quad \equiv$

fix $x_1 \; \ldots \; x_k$
assume $\underbrace{C_i\!:}_{\text{label}} \; \underbrace{term = (C_i \; x_1 \; \ldots \; x_k)}_{\text{formula}}$

Isar_Induction_Demo.thy

Structural induction for nat

Structural induction for nat

show $P(n)$
proof $(induction\ n)$
  case 0 $\equiv$ **let** $?case = P(0)$
  ⋮
  show $?case$
next
  case $(Suc\ n)$ $\equiv$ **fix** n **assume** $Suc\colon P(n)$
  let $?case = P(Suc\ n)$
  ⋮
  ⋮
  show $?case$
qed

Structural induction with $\Longrightarrow$

show $A(n) \Longrightarrow P(n)$		
proof $(induction\ n)$		
case $\mathit{0}$	$\equiv$	**assume** $\mathit{0}$: $A(\mathit{0})$
$\vdots$		**let** $?case = P(\mathit{0})$
show $?case$		
next		
case $(Suc\ n)$	$\equiv$	**fix** n
$\vdots$		**assume** Suc: $A(n) \Longrightarrow P(n)$ $A(Suc\ n)$
$\vdots$		**let** $?case = P(Suc\ n)$
show $?case$		
qed		

Named assumptions

In a proof of

$$A_1 \Longrightarrow \ldots \Longrightarrow A_n \Longrightarrow B$$

by structural induction:
In the context of

case C

we have

$C.IH$	the induction hypotheses
$C.prems$	the premises A_i
C	$C.IH + C.prems$

A remark on style

- **case** $(Suc\ n)$ … **show** $?case$
  is easy to write and maintain
- **fix** n **assume** $formula$ … **show** $formula'$
  is easier to read:
  - all information is shown locally
  - no contextual references (e.g. $?case$)

13 Proof by Cases and Induction

Rule Induction

Rule Inversion

Isar_Induction_Demo.thy

Rule induction

Rule induction

inductive $I :: \tau \Rightarrow \sigma \Rightarrow bool$
where
$rule_1$: $\ldots$
$\vdots$
$rule_n$: $\ldots$

show $I\ x\ y \Longrightarrow P\ x\ y$
proof $(induction\ rule{:}\ I.induct)$
  case $rule_1$
  $\ldots$
  show $?case$
next
$\vdots$
next
  case $rule_n$
  $\ldots$
  show $?case$
qed

Fixing your own variable names

case $(rule_i\ x_1\ \ldots\ x_k)$

Renames the first k variables in $rule_i$ (from left to right) to $x_1\ \ldots\ x_k$.

Named assumptions

In a proof of

$$I \ldots \Longrightarrow A_1 \Longrightarrow \ldots \Longrightarrow A_n \Longrightarrow B$$

by rule induction on $I \ldots$:
In the context of

case R

we have

$R.IH$	the induction hypotheses
$R.hyps$	the assumptions of rule R
$R.prems$	the premises A_i
R	$R.IH + R.hyps + R.prems$

13 Proof by Cases and Induction

Rule Induction

Rule Inversion

Rule inversion

inductive $ev :: nat \Rightarrow bool$ **where**
$ev0$: $ev\ 0$ |
$evSS$: $ev\ n \Longrightarrow ev(Suc(Suc\ n))$

What can we deduce from $ev\ n$?
That it was proved by either $ev0$ or $evSS$!

$$ev\ n \Longrightarrow n = 0 \lor (\exists k.\ n = Suc\ (Suc\ k) \land ev\ k)$$

Rule inversion = case distinction over rules

`Isar_Induction_Demo.thy`

Rule inversion

Rule inversion template

from $`ev\ n`$ **have** P
proof $cases$
  case $ev0$ $\qquad\qquad n = 0$
  $\vdots$
  show $?thesis$ $\ldots$
next
  case $(evSS\ k)$ $\qquad\qquad n = Suc\ (Suc\ k),\ ev\ k$
  $\vdots$
  show $?thesis$ $\ldots$
qed

Impossible cases disappear automatically