# Proof Reconstruction for Z3 in Isabelle/HOL

Sascha Böhme

Technische Universität München

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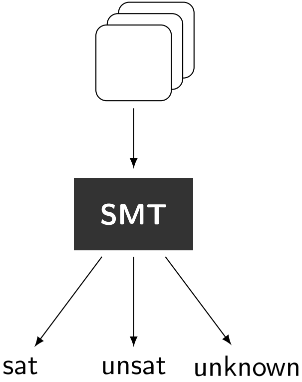
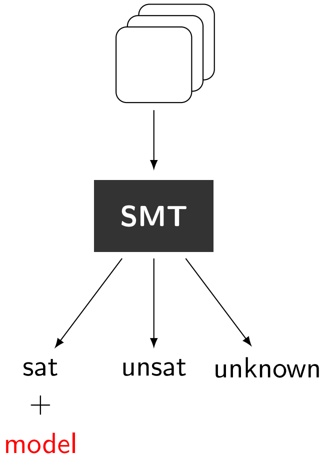



User perspective:

- SMT as “black-box” technology

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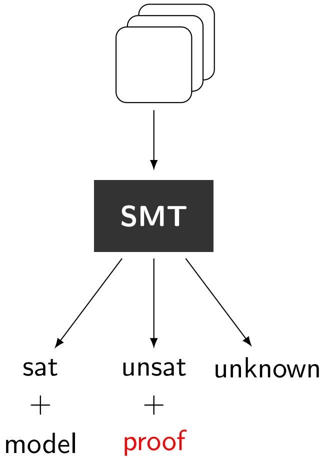


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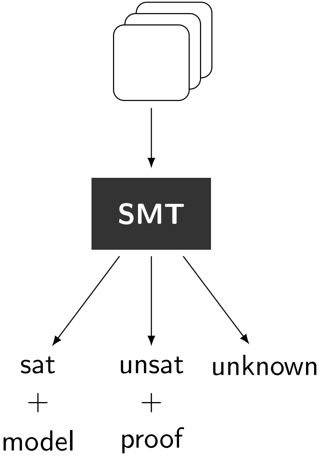


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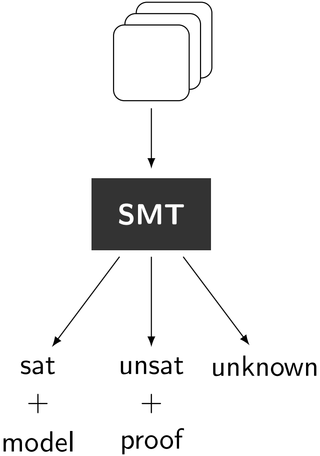
Increased confidence:

- checkable certificates

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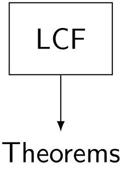
Our aim:

- certify proofs of Z3
- with Isabelle/HOL

# A Quick Glance at Isabelle/HOL




LCF kernel:

- abstract type: theorems
- operations: basic inference rules
- small

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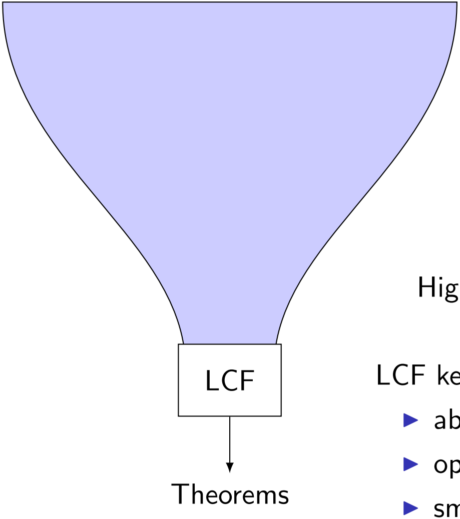


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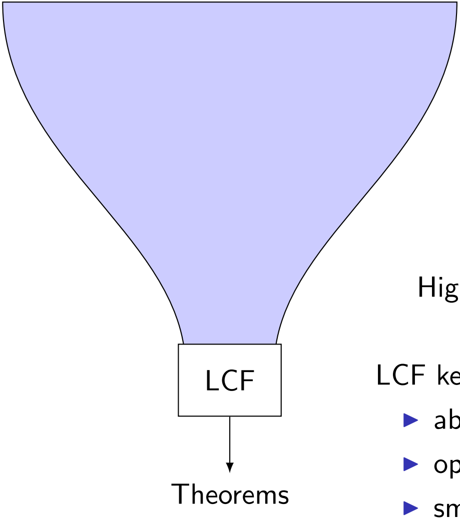


Proof tools:

- term rewriting (simplifier)
- tableaux prover (blast)
- decision procedures: linear arithmetic, quantifier elimination

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# Z3 Terms

Language: many-sorted first-order logic

Terms: $t$, $s$

- variables: $x$, $y$
- applications: $f \ t_1 \ldots t_n$
  - logical connectives: *true*, *false*, $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, $\sim$
- quantifiers: $\forall$, $\exists$
- terms of sort *bool* (formulas): $P$

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$$(\neg x \vee \mathit{false}) \sim (\neg y) \quad \equiv \quad (\exists x.\, \neg x \vee \mathit{false}) \leftrightarrow (\exists y.\, \neg y)$$

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Natural mapping into higher-order logics (Isabelle/HOL)

- equisatisfiability: representable as equivalence with one exception: Skolemization

# Z3 Proofs

Natural deduction style:

$$\frac{\Gamma_1 \vdash P_1 \qquad \Gamma_2 \vdash P_1 \leftrightarrow P_2}{\Gamma_1 \cup \Gamma_2 \vdash P_2}\ \mathbf{mp}_{\leftrightarrow}$$

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28 proof rules:

- core logic: **asserted**, **unit**, . . .
- equality: **refl**, **trans**, . . .
- quantifiers: **quant-inst**, **elim-unused**, . . .
- theories: **rewrite**, **th-lemma**

# Proof Reconstruction

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- one method for every rule

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- additional checks (for debugging)

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- Isabelle proof tools (simplifier, blast) (9 rules)
- specialized treatment (9 rules)
  - in some cases: optimizations

# Congruence

$$\frac{\Gamma_1 \vdash t_1 = s_1 \qquad \dots \qquad \Gamma_n \vdash t_n = s_n}{\bigcup_{i \leq n} \Gamma_i \vdash f\ t_1 \dots t_n = f\ s_1 \dots s_n}\ \textbf{mono}$$

In principle: provable by simplifier (term rewriting)

But: one of the central rules!

- optimization is worthwhile

Thus: combination of

- congruence: $f = g \Longrightarrow x = y \Longrightarrow f\ x = g\ y$
- reflexivity: $t = t$

# Skolemization

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Rewriting (**rewrite**):

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Theory reasoning (**th-lemma**):

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- linear arithmetics: Fourier-Motzkin elimination
- arrays: simplifier

# Experimental Results

- 5 SMT-LIB logics
- 100 unsatisifiably benchmarks (randomly selected)
- timeout: Z3: 2 minutes, Isabelle/HOL: 10 minutes

| Logic | Solved by Z3 | Reconstruction | | | Factor |
|---|---|---|---|---|---|
| | | Success | Failure | Timeout | |
| QF_UF | 96 | 33 | 27 | 36 | 6.5 |
| QF_UFLIA | 99 | 93 | 0 | 6 | 29.6 |
| QF_UFLRA | 100 | 43 | 0 | 57 | 558.3 |
| AUFLIA | 100 | 50 | 31 | 19 | 81.3 |
| AUFLIRA | 100 | 81 | 6 | 13 | 24.3 |

Bad performance:

- only few optimizations implemented
- huge formulas of benchmarks

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- Unit resolution:

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- Transitivity:

$$\frac{\Gamma_1 \vdash\ s = t \qquad \Gamma_2 \vdash\ u = t}{\Gamma_1 \cup \Gamma_2 \vdash\ s = u}\ \textbf{trans}$$

# Conclusion

Proof reconstruction for Z3:

- in Isabelle/HOL: certification by LCF kernel
- challenges: equisatisfiability, huge formulas
- helped to debug Z3 proof generation

Future work:

- improve performance
- integrate into Isabelle/HOL
- consider further theories