Automating Asymptotics in a Theorem Prover

Manuel Eberl

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Formal Methods in Mathematics
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My Christmas Project

I found some lovely 5-pages of lecture notes on Transcendental Number Theory by Filaseta:

4 The Irrationality of $\zeta(3)$

For $s > 1$, we define $\zeta(s) = \sum_{n=1}^{\infty} 1/n^s$. We give here a proof by Frits Beukers that $\zeta(3)$ is irrational (the result itself being originally due to R. Apery).

Theorem 10. *The number $\zeta(3) = \sum_{n=1}^{\infty} 1/n^3$ is irrational.*

In addition to Lemma 1 of the previous section (and the notation given there), we make use of the following results.

Lemma 2. *Let r and s be nonnegative integers. If $r > s$, then*

$$\int_0^1 \int_0^1 -\frac{\log(xy)}{1-xy} x^r y^s \, dx \, dy$$

is a rational number whose denominator when reduced divides d_r^3. Also,

$$\int_0^1 \int_0^1 -\frac{\log(xy)}{1-xy} x^r y^r \, dx \, dy = 2\left(\zeta(3) - \sum_{k=1}^{r} \frac{1}{k^3}\right).$$

Proof. Integrating by parts, we obtain that for $k \geq 0$

$$\int_0^1 (\log x) x^{r+k} \, dx = \lim_{\epsilon \to 0} \int_\epsilon^1 (\log x) x^{r+k} \, dx$$

My Christmas Project

So I decided to formalise them:

The Irrationality of $\zeta(3)$

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December 28, 2019

Abstract

This article provides a formalisation of Beukers's straightforward analytic proof [2] that $\zeta(3)$ is irrational. This was first proven by Apéry [1] (which is why this result is also often called 'Apéry's Theorem') using a more algebraic approach. This formalisation follows Filaseta's presentation of Beukers's proof [5].

Contents

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Mathematical proofs in a proof assistant (compared to pen-and-paper)

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But I want to talk about one in particular.

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  Most mathematical results have not been formalised
  And even if: perhaps not in the system you use.

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The reader is *the proof assistant*.

Domain-Specific Automation

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For effective formalisation of mathematics, we need to *teach* proof assistants these skills.

Examples for Domain-Specific Automation

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- Evaluating winding numbers (Li)
- Real asymptotics (E.)

Automating Real Asymptotics in Isabelle/HOL

- Interactive theorem prover; mostly *Higher Order Logic*

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- Archive of Formal Proofs:
  Large collection of Isabelle proof developments

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If you have to do this every 5 minutes, it gets annoying.

Example: Stieltjes constants

$$\gamma_n = \sum_{k=1}^{\infty} \left(\frac{\ln^n k}{k} - \frac{\ln^{n+1}(k+1) - \ln^{n+1} k}{n+1} \right)$$

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But proving those asymptotics by hand is a lot of work.

Example: Lemma required for Akra–Bazzi

$$\lim_{x\to\infty}\left(1-\frac{1}{b\log^{1+\varepsilon}x}\right)^p\left(1+\frac{1}{\log^{\varepsilon/2}\left(bx+\frac{x}{\log^{1+\varepsilon}x}\right)}\right)-\left(1+\frac{1}{\log^{\varepsilon/2}x}\right) \;=\; 0^+$$

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Original author: ‘Trivial, just Taylor-expand it!’

In Isabelle:

lemma akra_bazzi_aux:
  filterlim
    $(\lambda x.\ (1 - 1/(b * \ln x\,\hat{}\,(1+\varepsilon))\,\hat{}\,p) *$
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Omitted: 700 lines of messy proofs

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How does it work?

Multiseries Expansions

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Disclaimer: None of this was invented by me.

Related Work:

- *Asymptotic Expansions of exp–log Functions*
  by Richardson, Salvy, Shackell, van der Hoeven
- *On Computing Limits in a Symbolic Manipulation System*
  by Gruntz
- *Verified Real Asymptotics in Isabelle/HOL*
  by E.

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Power series expansions are insufficient for many important functions: $\exp(x)$, $\ln(x)$, $\Gamma(x)$ for $x \to \infty$

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- Typical basis: $\exp(x^2)$, $\exp(x)$, x, $\ln x$, $\ln\ln x$

A coalgebraic view of Multiseries

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&\quad \mathsf{Const}\ :\ \mathbb{R} \to \mathsf{MS}\ []\\
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- exp and ln at singular points require specialised procedures and may add new basis elements
- For operations like Γ, erf, li:
  factor out singularities and treat them separately

Connecting Series and Functions

For simple power series, $f \sim ts$ can be expressed coinductively:

$$\frac{f(x) \in O(x^e) \qquad f(x) - c\,x^e \;\sim\; ts}{f(x) \;\sim\; (c, e) :: ts}$$

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Operations are defined corecursively; correctness is proven coinductively. Both are straightforward.

The same works for multiseries quite similarly.

Finding Expansions

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End result: Theorem that $\sin(1/x) + \exp(x)$ has the following expansion w. r. t. basis $(\exp(x), x)$:

$$\begin{aligned} &\text{lift\_expansion } (\text{sin\_ms } (\text{Series } [(1, -1)])) + \\ &\quad \text{Series } [(\text{Series } [(1, 0)], 1)] \end{aligned}$$

which evaluates to

$$\exp(x) + x^{-1} - \frac{1}{6}x^{-3} + \frac{1}{120}x^{-5} - \dots$$

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- Both of these are difficult or even undecidable

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This works surprisingly well

Proof Method

With some pre-processing, we can automatically prove statements of the form

- $f(x) \longrightarrow c$
- $f(x) \sim g(x)$
- $f(x) < g(x)$ eventually
- $f(x) \in L(g(x))$ for any Landau symbol L

as $x \to l$ for $l \in \mathbb{R} \cup \{\pm\infty\}$

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- $f(x) \in L(g(x))$ for any Landau symbol L

as $x \to l$ for $l \in \mathbb{R} \cup \{\pm\infty\}$

f and g can be built from $+$ $-$ $\cdot$ $/$ $\ln$ $\exp$ $\min$ $\max$ $\hat{}$ $|\cdot|$ $\sqrt[n]{\cdot}$ without restrictions

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sin, cos, tan at finite points also possible.

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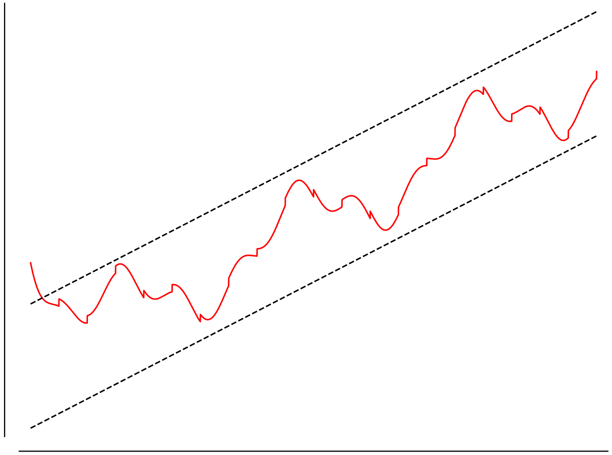

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- Most uses are for fairly trivial examples
- But: Some others would have been quite painful without the method.
- And: The benefit of not having to stop and think about trivialities like $x^2 - x \to \infty$ should not be underestimated!

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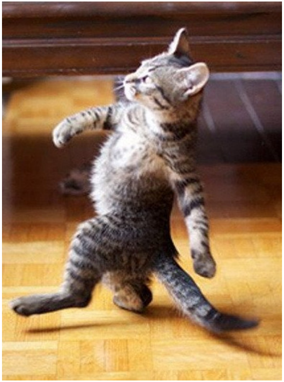

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- Laurent series expansions for complex functions
  $\Longrightarrow$ automatic computation of poles, residues, etc.

Questions? Demo?