# For a Few Dollars More

## Verified Fine-Grained Algorithm Analysis Down to LLVM

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April, 2021

## C++ Standard

### 25.4.1 Sorting [alg.sort]

#### 25.4.1.1 sort [sort]

```
template<class RandomAccessIterator>
  void sort(RandomAccessIterator first, RandomAccessIterator last);

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*Effects:* Sorts the elements in the range `[first,last)`.

*Requires:* `RandomAccessIterator` shall satisfy the requirements of `ValueSwappable` (17.6.3.2). The type of `*first` shall satisfy the requirements of `MoveConstructible` (Table 20) and of `MoveAssignable` (Table 22).

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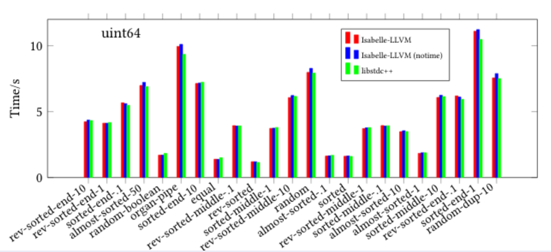


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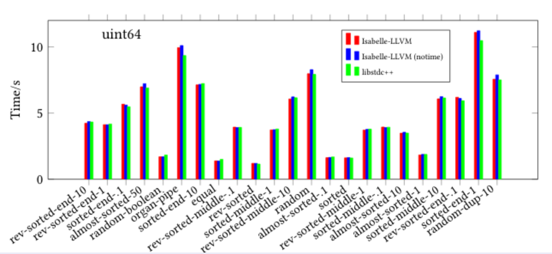


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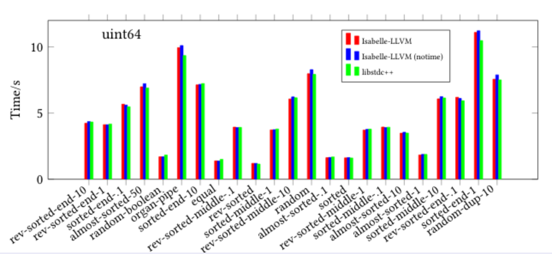


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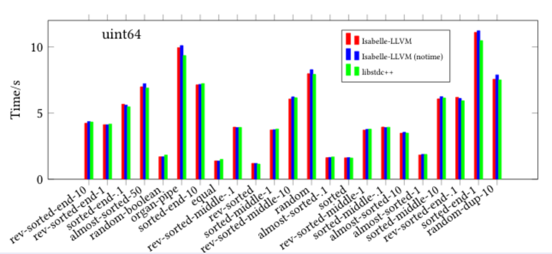


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# Top-Down Approach

- First verification of a competitive implementation of Introsort with Time Bound
- Stepwise Refinement Calculus with Resource Currencies
- Correctness-and-Time-Bound-Preserving Synthesis Mechanism
- LLVM Semantics with Cost Model
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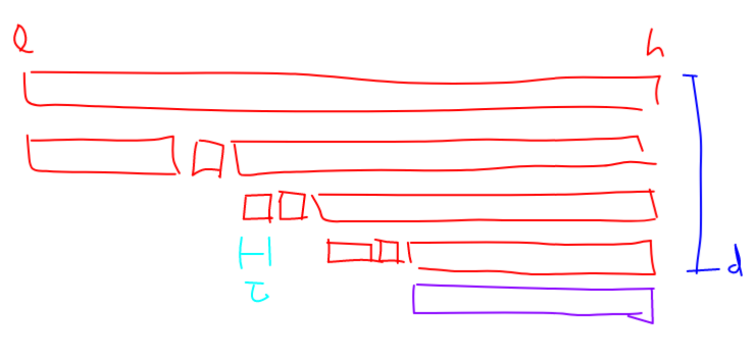
# Introsort

## Quicksort Scheme



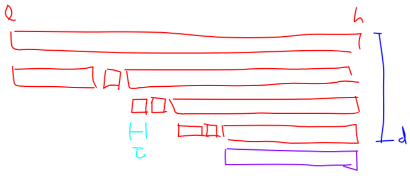
# Introsort

## Musser's Pseudocode

```
Algorithm Introsort(A, f, b)
                              Inputs: A, a random access data structure containing the sequence
                                 of data to be sorted, in positions A[f], ..., A[b − 1];
                                f, the first position of the sequence
                                b, the first position beyond the end of the sequence
                              Output: A is permuted so that A[f] ≤ A[f+1] ≤ ... ≤ A[b − 1]
    Introsort_loop(A, f, b, 2 * Floor_lg(b − f))
    Insertion_sort(A, f, b)

Algorithm Introsort_loop(A, f, b, depth_limit)
                              Inputs: A, f, b as in Introsort;
                                depth_limit, a nonnegative integer
                              Output: A is permuted so that A[i] ≤ A[j]
                                for all i, j: f ≤ i < j < b and size_threshold < j − i
    while b − f > size_threshold
        do if depth_limit = 0
                then Heapsort(A, f, b)
                    return
            depth_limit := depth_limit − 1
            p := Partition(A, f, b, Median_of_3(A[f], A[f+(b−f)/2], A[b−1]))
            Introsort_loop(A, p, b, depth_limit)
            b := p
```

## Quicksort Scheme



```
introsort xs l h =
  n ← return h−l;                   ($_sub)
  if_c n > 1 then                   ($_lt)
    xs ← almost_sort_spec xs l h;   ($_almost_sort)
    xs ← final_sort_spec xs l h;    ($_final_sort)
    return xs
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- $E_1 \; \mathit{slice\_sort} = \$_{sub} + \$_{lt} + \$_{if} + \$_{almost\_sort} + \$_{final\_sort}$

```
introsort_rec xs l h d =
  assert (l ≤ h);
  n ← h − l;                                  ($_sub)
  if_c n > τ then                              ($_lt)
    if_c d = 0 then                            ($_eq)
      slice_sort_spec xs l h                   ($_sort_c (μ (h-l)))
    else
      (xs, m) ← partition_spec xs l h;         ($_partition_c (h-l))
      d′ ← d − 1;                              ($_sub)
      xs ← introsort_rec xs l m d′;
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- $introsort\_rec \leq \Downarrow_C \; E_2 \; (almost\_sort_{spec} \; \$_{almost\_sort})$

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- $E_2\ almost\_sort = introsort\_rec_{cost}\ (h - l,\ d)$

# Stepwise Refinement

- Refine $\mathit{slice\_sort}_{spec}$ with HEAPSORT in $O(n \log n)$
- Refine $\mathit{partition}_{spec}$ in $O(n)$
- Refine $\mathit{final\_sort}_{spec}$ with INSERTIONSORT in $O(\tau\ n)$
  - Unguarded Optimization
- Synthesis to LLVM Code

# Final result

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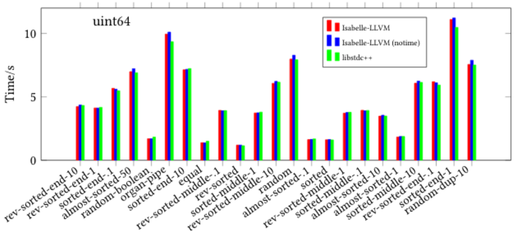

# In the Paper

- Nondeterministic Result Monad with Time (NREST)
- Refinement Patterns + Automation
- Synthesis Mechanism (Connecting NREST with LLVM Monad)
- LLVM Semantics + Cost Model
    - Basic Reasoning Infrastructure

# Conclusion

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Formalization & Benchmarks & More:
https://www21.in.tum.de/~haslbema/llvm-time/

Thank you!