Automata and Formal Languages II

Tree Automata

Peter Lammich

SS 2015

Overview by Lecture

- Apr 14: Slide 3
- Apr 21: Slide 2
- Apr 28: Slide 4
- May 5: Slide 50
- May 12: Slide 56
- May 19: Slide 64
- May 26: Holiday
- Jun 02: Slide 79
- Jun 09: Slide 90
- Jun 16: Slide 106
- Jun 23: Slide 108
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- Jul 14: Slide 148

Organizational Issues

Lecture Tue 10:15 – 11:45, in MI 00.09.38 (Turing)

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Conflict with Equational Logic.

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- Application: Analysis of Concurrent Programs
  - Dynamic Pushdown Networks (DPN)

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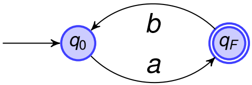

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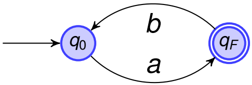

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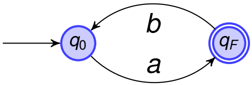

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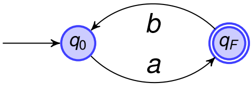

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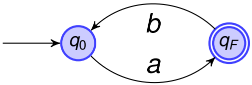

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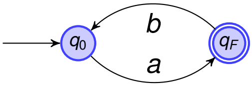

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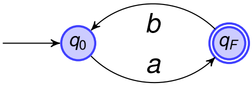

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```
    a
   / \
  b   b
 / \ / \
a  a a  a
/\ /\ /\ /\
LLLLLLLL
```

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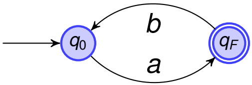

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```
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   / \         / \
  b   b       b   L
 / \ / \     / \
a a a a     a a
/\ /\ /\ /\ /\ /\
LLLLLLLL    LLLL
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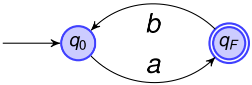

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```
     a              a
    / \            / \
  b     b        b     L
 / \   / \      / \
a  a  a  a     a  a
/\ /\ /\ /\    /\ /\
LLLLLLLL       LLLL
```

- We also write trees as terms
  - $a(b(a(L, L), a(L, L)), b(a(L, L), a(L, L)))$
  - $a(b(a(L, L), a(L, L)), L)$

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- Ground terms: $T(\mathcal{F}) := T(\mathcal{F}, \emptyset)$. Terms without variables.

Examples

- We also write a ranked alphabet as $\mathcal{F} = f_1/a_1, f_2/a_2, \ldots, f_n/a_n$, meaning $\mathcal{F} = (\{f_1, \ldots, f_n\}, (f_1 \mapsto a_1, \ldots, f_n \mapsto a_n))$

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    - We will use infix-notation for terms when appropriate

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   /   \
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```
      +
     / \
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   |   / \
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      |
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- Intuition: Use the rules from Δ to re-write a given tree to a final state
- For a tree $t \in T(\mathcal{F})$ and a state q, we define $t \to_{\mathcal{A}} q$ as the least relation that satisfies

$$f(q_1, \ldots, q_n) \to q \in \Delta, \forall 1 \leq i \leq n.\ t_i \to_{\mathcal{A}} q_i \implies f(t_1, \ldots, t_n) \to_{\mathcal{A}} q$$

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  - $t \to_{\mathcal{A}} q$: Tree t is *accepted* in state q

Tree Automata

- A (nondeterministic) finite tree automaton (NFTA) over alphabet $\mathcal{F}$ is a tuple $\mathcal{A} = (Q, \mathcal{F}, Q_f, \Delta)$ where
  - Q is a finite set of *states*. $Q \cap F_0 = \emptyset$
  - $Q_f \subseteq Q$ is a set of *final states*
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- The language $L(\mathcal{A})$ of $\mathcal{A}$ are all trees accepted in final states

  $$L(\mathcal{A}) := \{t \mid \exists q \in Q_f.\ t \to_{\mathcal{A}} q\}$$

Example

- Tree automaton accepting arithmetic expressions that evaluate to even numbers

$$
\begin{array}{llll}
\mathcal{F} = 0/0, Suc/1, +/2 & & & \\
Q := \{e, o\} & Q_f = \{e\} & & \\
0 \to e & Suc(e) \to o & Suc(o) \to e & \\
e + e \to e & e + o \to o & o + e \to o & o + o \to e
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- Examples for runs on board
  - $Suc(Suc(0)) + Suc(0) + Suc(0)$
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Remark

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- On board: Accepting [], and $[0, Suc(0)]$

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- From now on, we assume tree automata without ϵ-rules, unless noted otherwise.

Last Lecture

- Nondeterministic Finite Tree Automata (NFTA)
  - Ranked alphabet, Terms/Trees
  - Rules: $f(q_1, \ldots, q_n) \rightarrow q$
  - Intuition: Rewrite tree to single state
- Epsilon rules
  - $q \rightarrow q'$
  - Do not increase expressiveness (recognizable languages)

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- $\mathcal{A}$ is *reduced*, if all states in Q are accessible.

Membership Test for DFTA

- Complete DFTAs have a simple (and efficient) membership test

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  let
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- Note: For NFTAs, we need to backtrack, or use on-the-fly determinization

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- Theorem follows from this.

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$\qquad s = \{q \in Q \mid \exists q_1 \in s_1, \dots, q_n \in s_n.\ f(q_1, \dots, q_n) \to q \in \Delta\}$
until No more rules can be added to Δ_d
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Examples

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- But, exponential blowup not avoidable in general

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Let L be a regular language. Then, there is a constant $k > 0$ such that for every $t \in L$ with $Height(t) > k$, there is a context C, a non-trivial context C', and a term u such that

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    - Choose $n = |C| + |u| - 1$ to show that this is not prime for all n

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     - $\Longleftarrow$: Pump with infinitely many n

Last Lecture

- Deterministic Automata
  - Powerset construction
- Pumping Lemma

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Closure Properties

Theorem

- *The class of regular languages is closed under union, intersection, and complement.*
- *Automata for union, intersection, and complement can be computed.*

Union

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  - Intuition: Recognize with both automata in parallel.

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  - $\mathcal{A}$ is deterministic/complete if $\mathcal{A}_1$ and $\mathcal{A}_2$ are.
- Product construction can also be combined with reduction algorithm, to avoid construction of inaccessible states.

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- For NFTA: Polynomial time intersection, union. Exp-time complement.

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- Emptiness check: Time $O(|\mathcal{A}|)$. Exercise!

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- $h_{\mathcal{F}}$ determines a *tree homomorphism* $h : T(\mathcal{F}) \to T(\mathcal{F}')$:

$$h(f(t_1, \ldots, t_n)) := h_{\mathcal{F}}(f)(x_1 \mapsto h(t_1), \ldots, x_n \mapsto h(t_n))$$

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- Proof idea: For each original rule $f(q_1, \ldots, q_n)$, insert rules that recognize $h_{\mathcal{F}}[q_1, \ldots, q_n]$

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- $Pos(t)$ is the set of valid positions in t

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    - $q^r_\varepsilon \to q \in \Delta'$

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  - Formally: Induction on size of derivation $t \rightarrow_{\mathcal{A}'} q$

Last lecture

- Closure properties: Union, intersection, complement
- Tree homomorphisms
  - Idea: Replace node by tree with „holes"
  - $\mathit{and}(x_1, x_2) \mapsto \mathit{not}(\mathit{or}(\mathit{not}(x_1), \mathit{not}(x_2)))$
- Regular languages closed under *linear* homomorphisms
  - Linear: No subtrees are duplicated

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- Common technique to show regularity/decidability
    - Can be generalized to (macro) tree transducers

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- This is obviously a generalization of the acceptance relation we defined earlier

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Last Lecture

- Inverse homomorphisms preserve regularity
- Started Myhill-Nerode Theorem

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- An equivalence relation is of *finite index*, if there are only finitely many equivalence classes

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- An equivalence relation $\equiv$ on $T(\mathcal{F})$ is a *congruence*, iff

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  - Intuition: Functions are equivalent if applied to equivalent arguments.

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Myhill-Nerode Theorem

Theorem

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Convention

- Complete DFTAs are written as $(Q, \mathcal{F}, Q_f, \delta)$
  - with $\delta : (\mathcal{F}_n \times Q^n \to Q)_n$
  - Corresponds to Δ via

$$f(q_1, \dots, q_n) \to q \text{ iff } \delta(f, q_1, \dots, q_n) = q$$

  - Naturally extended to trees

$$\delta(f(t_1, \dots, t_n) = \delta(f, \delta(t_1), \dots, \delta(t_n))$$

  - Compatible with $\to_{\mathcal{A}}$, i.e.

$$t \to_{\mathcal{A}} q \text{ iff } \delta(t) = q$$

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- The DFTA $\mathcal{A}_{min} := (Q_{min}, \mathcal{F}, Q_{min_f}, \Delta_{min})$ recognizes the language L

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    - This mapping is consistent and bijection

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- $L(\mathcal{A}_{min}) = L(\mathcal{A})$. Proof on board.

Last Lecture

- Myhill-Nerode Theorem
- Minimization of tree automata

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- The language of $\mathcal{A}$ is $L(\mathcal{A}) := \{t \mid \exists q \in I.\ q \to_{\mathcal{A}} t\}$

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- Unfortunately: There are regular languages not accepted by any deterministic top-down FTA
  - $L = \{f(a, b), f(b, a)\}$. Obviously regular. Even finite.
  - But: Any deterministic top-down FTA that accepts the words in L also accepts $f(a, a)$.

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    - Cf.: Elimination of epsilon rules

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- Correctness (Ideas)

Normalized Regular Tree Grammars

- RTG is normalized, iff all productions have the form $n \to f(n_1, \dots, n_n)$ for $n, n_1, \dots, n_n \in N$
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Normalized RTGs and top-down NTFAs

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Normalized RTGs and top-down NTFAs

- Obviously, normalized RTGs are isomorphic to top-down NTFAs
- Thus, exactly the regular languages can be expressed by RTGs

Theorem

A language is regular if and only if it can be described by a regular tree grammar.

Last Lecture

- Myhill Nerode Theorem
- Minimization Algorithm
- Top-Down Tree Automata
- Regular Tree Grammars
- Started: Tree Regular Expressions

Table of Contents

Recall: Word regular expressions

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- And generalize this to languages

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- And define concatenation

$$L_1 \cdot_i L_2 := L_1\{\Box_i \leftarrow L_2\}$$

Iteration

- Iteration $L^{n,i}$

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- Closure L^{*_i}

$$L^{*_i} := \bigcup_{n \in \mathbb{N}} L^{n,i}$$

Preservation of Regularity (Concatenation)

Theorem

Substitution preserves regularity, i.e., let $L, L_1, \ldots, L_n$ be regular languages, then $L' := L\{a_1 \leftarrow L_1, \ldots, a_n \leftarrow L_n\}$ is a regular language

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- Corollary: Concatenation preserves regularity, i.e., for regular languages L_1, L_2, the language $L_1 \cdot L_2$ is regular.

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Closure preserves regularity, i.e., let L be a regular language. Then, L^* is a regular language.

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  - $L(G') \subseteq L^*$: Re-ordering derivation. Formally: Induction on derivation length.

Tree Regular Expressions

- Syntax

$$e ::= \emptyset \mid f(\underbrace{e, \ldots, e}_{n \text{ times}}) \text{ for } f \in \mathcal{F}_n \mid e + e \mid e \cdot_i e \mid e^{*_i}$$

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- Semantics

$$\begin{aligned}
[\![\emptyset]\!] &= \emptyset \\
[\![f(e_1, \ldots, e_n)]\!] &= \{f(t_1, \ldots, t_n) \mid t_i \in [\![e_i]\!]\} \\
[\![e_1 + e_2]\!] &= [\![e_1]\!] \cup [\![e_2]\!] \\
[\![e_1 \cdot_i e_2]\!] &= [\![e_1]\!] \cdot_i [\![e_2]\!] \\
[\![e_1^{*_i}]\!] &= [\![e_1]\!]^{*_i}
\end{aligned}$$

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Theorem

A tree language L is regular if and only if there is a regular expression e with $L = [\![e]\!]$

- Proof ($\Longleftarrow$): Straightforward, by induction on e, using preservation of regularity by union, concatenation, and closure

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- Proof ($\Longrightarrow$): Construct reg-exp inductively over increasing number of states

Kleene Theorem for Tree Languages (Proof)

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- For $j > 0$:

$$T(i,j,K) = \underbrace{T(i, j-1, K \cup \{q_j\})}_{\text{Initial segment}} \cdot_{q_j} \underbrace{T(j, j-1, K \cup \{q_j\})^{*, q_j}}_{\text{Runs between } q_j\text{s}} \cdot_{q_j} \underbrace{T(j, j-1, K)}_{\text{Final segment}}$$

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- Regular expression for $L(\mathcal{A})$ can be constructed

Last Lecture

- Tree regular expressions
- Kleene theorem
  - Tree regular expressions can express exactly the tree regular languages

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Program Analysis

- Theorem of Rice: Properties of programs undecidable
- Need approximations
- Standard approximation: Ignore branching conditions
  - **if** (b) ... **else** ... Consider both branches, independent of *b*
  - Nondeterministic program

Attack Plan

- Properties: Reachability of configuration/regular set of configurations
- First, consider programs with recursion
  - Modeled by pushdown systems (PDS)
- Then, add process creation
  - Modeled by dynamic pushdown systems (DPN)
- Then synchronization through well-nested locks
  - DPN with locks

Recursion

- If program has no procedures
  - Runs can be described by word automaton
  - Example on board
- If program has procedures
  - Runs can be described by push-down system (PDS)

Example

```
    void p() {
1:    if (...) p() else return;
2:    x=y;
3:    return;
    }
```

$1 \overset{\tau}{\hookrightarrow} 12$

$1 \overset{\tau}{\hookrightarrow} \varepsilon$

$2 \overset{x=y}{\hookrightarrow} 3$

$3 \overset{\tau}{\hookrightarrow} \varepsilon$

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Push-Down Systems (PDS)

- In order to model (finitely many) return values, we add state
- A *push-down system* (PDS) M is a tuple $(P, \Gamma, \mathrm{Act}, p_0, \gamma_0, \Delta)$ where
  - P is a finite set of states
  - Γ is a finite stack alphabet
  - Act is a finite set of actions
  - $p_0\gamma_0 \in P\Gamma$ is the initial configuration
  - Δ is a finite set of rules, of the form

$$p\gamma \overset{a}{\hookrightarrow} p'w \text{ where } p, p' \in P,\ a \in \mathrm{Act},\ \gamma \in \Gamma, \text{ and } w \in \Gamma^*$$

PDS - Semantics

- Configurations have the form $pw \in P\Gamma^*$
- The step-relation $\rightarrow \subseteq P\Gamma^* \times \mathrm{Act} \times P\Gamma^*$ is defined by

$$p\gamma w \xrightarrow{a} p'w'w \text{ if } p\gamma \overset{a}{\hookrightarrow} p'w' \in \Delta$$

- $\rightarrow^* \subseteq P\Gamma^* \times \mathrm{Act}^* \times P\Gamma^*$ is its extension to sequences of steps
  - $pw \xrightarrow{l}{}^* p'w'$ iff $l = a_1 \ldots a_n$ and $pw \overset{a_1}{\hookrightarrow} \ldots \overset{a_n}{\hookrightarrow} p'w'$

Normalized PDS

- Simplifying assumptions
  - There are only three types of rules

$$p\gamma \overset{a}{\hookrightarrow} p'\gamma' \qquad \text{for } p, p' \in P \text{ and } \gamma, \gamma' \in \Gamma \qquad \text{(base)}$$

$$p\gamma \overset{a}{\hookrightarrow} p'\gamma_1\gamma_2 \qquad \text{for } p, p' \in P \text{ and } \gamma, \gamma_1, \gamma_2 \in \Gamma \qquad \text{(call)}$$

$$p\gamma \overset{a}{\hookrightarrow} p' \qquad \text{for } p, p' \in P \text{ and } \gamma \in \Gamma \qquad \text{(return)}$$

    - Does not reduce expressiveness. Emulate rule $p\gamma \overset{\gamma}{\hookrightarrow}_1 \dots \gamma_n$ by sequence of call rules.
  - The empty stack must not be reachable
    - Does not reduce expressiveness
    - Introduce fresh $\perp$ stack symbol, a rule $p_0\perp \overset{\tau}{\hookrightarrow} p_0\gamma_0\perp$, and set initial state to $p_0\perp$
    - τ models an action that has no effect (skip)
- From now on, we assume that PDS are normalized

Execution Trees

- Model executions of PDS as tree
  - Also incomplete executions, i.e., execution may stop everywhere
  - This describes all reachable configurations
- A node represents a step
- If a call returns, the call-node has two successors
  - Left successor describes execution of procedure
  - Right successor describes execution of remaining program
- Execution trees described by the following tree grammar

$$XR ::= \langle Base \rangle(XR) \mid \langle Call \rangle^R(XR, XR) \mid \langle Return \rangle$$

$$XN ::= \langle Base \rangle(XN) \mid \langle Call \rangle^N(XN) \mid \langle Call \rangle^R(XR, XN) \mid \langle P \times \Gamma \rangle$$

  - Where *Base*, *Call*, *Return* are rules of respective type
  - Intuition: XR – Returning execution trees, XN – non-returning execution trees

Example

$p1 \overset{\tau}{\hookrightarrow} p12$

$p2 \overset{x=y}{\hookrightarrow} p3$

$p3 \overset{\tau}{\hookrightarrow} p$

$p1 \overset{\tau}{\hookrightarrow} p$

- Example execution tree
  - $\langle p1 \overset{\tau}{\hookrightarrow} p12\rangle^N(\langle p1 \overset{\tau}{\hookrightarrow} p12\rangle^R(\langle p1 \overset{\tau}{\hookrightarrow} p\rangle, \langle p2 \overset{x=y}{\hookrightarrow} p3\rangle(\langle p3\rangle)))$

Execution Trees of PDS

- Execution trees of PDS $M = (P, \Gamma, \mathrm{Act}, p_0, \gamma_0, \Delta)$ described by tree automata $\mathcal{A}_M = (Q, \mathcal{F}, I, \Delta_{\mathcal{A}_M})$
- States: $Q = P\Gamma \cup P\Gamma|P$
  - $p\gamma$ – produce non-returning execution trees (from XN)
  - $p\gamma|p''$ – produce execution trees that return to state p'' (from XR)
  - Initial state: $I = \{p_0\gamma_0\}$
- Rules

$$
\begin{array}{ll}
p\gamma \to \langle p\gamma \overset{a}{\hookrightarrow} p'\gamma' \rangle (p'\gamma') & \text{if } p\gamma \overset{a}{\hookrightarrow} p'\gamma' \in \Delta \\
p\gamma \to \langle p\gamma \overset{a}{\hookrightarrow} p'\gamma_1\gamma_2 \rangle^N (p'\gamma_1) & \text{if } p\gamma \overset{a}{\hookrightarrow} p'\gamma_1\gamma_2 \in \Delta \\
p\gamma \to \langle p\gamma \overset{a}{\hookrightarrow} p'\gamma_1\gamma_2 \rangle^R (p'\gamma_1|p'', p''\gamma_2) & \text{if } p'' \in P \text{ and } p\gamma \overset{a}{\hookrightarrow} p'\gamma_1\gamma_2 \in \Delta \\
p\gamma \to \langle p\gamma \rangle & \\
p\gamma|p'' \to \langle p\gamma \overset{a}{\hookrightarrow} p'\gamma' \rangle (p'\gamma'|p'') & \text{if } p\gamma \overset{a}{\hookrightarrow} p'\gamma' \in \Delta \\
p\gamma|p'' \to \langle p\gamma \overset{a}{\hookrightarrow} p'\gamma_1\gamma_2 \rangle^R (p'\gamma_1|p''', p'''\gamma_2|p'') & \text{if } p''' \in P \text{ and } p\gamma \overset{a}{\hookrightarrow} p'\gamma_1\gamma_2 \in \Delta \\
p\gamma|p'' \to \langle p\gamma \overset{\tau}{\hookrightarrow} p'' \rangle & \text{if } p\gamma \overset{\tau}{\hookrightarrow} p'' \in \Delta
\end{array}
$$

Execution Trees – Intuition of rules

- $p\gamma \to \langle p\gamma \overset{a}{\hookrightarrow} p'\gamma' \rangle (p'\gamma')$ (Base)
  - Make a base step, then continue execution from $p'\gamma'$
- $p\gamma \to \langle p\gamma \overset{a}{\hookrightarrow} p'\gamma_1\gamma_2 \rangle^N (p'\gamma_1)$ (Call, no-return)
  - Continue execution from $p'\gamma_1$.
  - As call does not return, γ_2 is never looked at again, and remaining execution does not depend on it
- $p\gamma \to \langle p\gamma \overset{a}{\hookrightarrow} p'\gamma_1\gamma_2 \rangle^R (p'\gamma_1 | p'', p''\gamma_2)$ (Call, return)
  - Execute procedure, it returns with state p''. Then continue execution from $p''\gamma_2$.
- $p\gamma \to \langle p\gamma \rangle$ (Finish)
  - Non-deterministically decide that execution ends here
- $p\gamma | p'' \to \langle p\gamma \overset{a}{\hookrightarrow} p'\gamma' \rangle (p'\gamma' | p'')$ (Base)
  - Base step, then continue execution
- $p\gamma | p'' \to \langle p\gamma \overset{a}{\hookrightarrow} p'\gamma_1\gamma_2 \rangle^R (p'\gamma_1 | p''', p'''\gamma_2 | p'')$ (Call, return)
  - Return from called procedure in state p''', then continue execution
- $p\gamma | p'' \to \langle p\gamma \overset{\tau}{\hookrightarrow} p'' \rangle$ (Return)
  - Return rule returns to specified state p''

Reached Configuration

- Function $c : XN \to P\Gamma$ extracts reached configuration from execution tree

$$
\begin{aligned}
c(\langle p\gamma \overset{a}{\hookrightarrow} p'\gamma'\rangle(t)) &= c(t) \\
c(\langle p\gamma \overset{\tau}{\hookrightarrow} p'\gamma_1\gamma_2\rangle^R(t_1, t_2)) &= c(t_2) \\
c(\langle p\gamma \overset{\tau}{\hookrightarrow} p'\gamma_1\gamma_2\rangle^N(t)) &= c(t)\gamma_2 \\
c(\langle p\gamma\rangle) &= p\gamma
\end{aligned}
$$

- Side note: This is a tree to string transducer
  - Thus, set of execution trees that reach a regular set of configurations is regular

Last Lecture

- Pushdown systems
  - Configuration $pw \in P\Gamma^*$
  - Semantics by step relation
- Execution trees
  - Intuition: Node for steps. Returning call nodes are binary.
  - Set of execution trees of PDS is regular
  - Mapping of execution tree to reached configuration
- Correlation:
  - Reachable configurations wrt. step relation and execution trees match

Relating Execution Trees and PDS Semantics

Theorem

Let M be a PDS. Then $\exists l.\ p_0\gamma_0 \xrightarrow{l}{}^{*} p'w$ *iff* $\exists t.\ t \in L(\mathcal{A}_M) \wedge c(t) = p'w$

- Note, a more general theorem would also relate the sequence of actions *l* and the execution tree
  - Proof ideas are the same

Last Lecture

- Proof of relation between execution trees and PDS semantics

Proof Outline

- Prove, for returning executions: $\exists l.\ p\gamma \xrightarrow{l}{}^{*} p''$ iff $\exists t.\ p\gamma | p'' \rightarrow t$
  - As c ignores returning executions, this simple statement is enough
- Prove, for non-returning executions:

$$\exists l.\ p\gamma \xrightarrow{l}{}^{*} p'w \wedge w \neq \varepsilon \text{ iff } \exists t.\ p\gamma \rightarrow t \wedge c(t) = p'w$$

- Main lemmas that are required
  - An execution can be repeated when we append some symbols to the stack:

$$\text{lemma stack-append: } pw \xrightarrow{l}{}^{*} p'w' \implies pwv \xrightarrow{l}{}^{*} p'w'v$$

  - If we have an execution, the topmost stack-symbol is either popped at some point, or the execution does not depend on the stack below the topmost symbol. Lemma return-cases:

$$p\gamma w \xrightarrow{l}{}^{*} p'w' \implies$$

$$\exists p''\ l_1\ l_2.\ p\gamma \xrightarrow{l_1}{}^{*} p'' \wedge p''w \xrightarrow{l_2}{}^{*} p'w' \wedge l = l_1 l_2 \qquad \text{(ret)}$$

$$\vee\ \exists w''.\ w' = w''w \wedge w'' \neq \varepsilon \wedge p\gamma \xrightarrow{l}{}^{*} p'w'' \qquad \text{(no-ret)}$$

  - Corollary: On a returning execution, we can find the point where the topmost stack symbol is popped

$$\text{lemma find-return: } p\gamma w \xrightarrow{l}{}^{*} p' \implies \exists l_1\ l_2\ p''.\ p\gamma \xrightarrow{l_1}{}^{*} p'' \wedge p''w \xrightarrow{l_2}{}^{*} p'$$

Proofs:

- On board
  - lemma return-cases (find-return is corollary)
  - Proofs for returning and non-returning executions

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Thread Creation

- Concurrent programs may create threads
- These run in parallel

Example

```
void p () {
  if (...) {
    spawn p;
    p();
  }
}

main () {
  p();
}
```

Dynamic Pushdown Networks

- Pushdown systems

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\begin{aligned}
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- Assumption: Empty stack not reachable in any spawned thread

Configurations

- Configurations are trees over the alphabet $\langle \mathit{pw} \rangle/1 \mid \mathit{Cons}/2 \mid \mathit{Nil}/0$

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- Convention: We identify c with the singleton list $Cons(c, Nil)$, and use $l_1 l_2$ for the concatenation of l_1 and l_2.
  - We may use $[c_1, \ldots, c_n]$ for the list $Cons(c_1, Cons(\ldots, Cons(c_n, Nil)\ldots)$ for clarification of notation.

Last Lecture

- Finished proof: Relation of execution trees and PDS semantics
- DPN (PDS + Thread creation)
- DPN-Semantics:
  - Configuration are trees, each node holds PDS-configuration (state+stack)
  - Children are threads that have been spawned by parent
- Extract reached configuration from execution tree

Semantics

- A step modifies a thread's state according to a rule

$$
\begin{aligned}
& C[\langle p\gamma w\rangle(I)] \overset{a}{\to} C[\langle p' w' w\rangle(I)] \\
& \qquad\qquad \text{if } p\gamma \overset{a}{\hookrightarrow} p' w' \in \Delta && \text{(no-spawn)} \\
& C[\langle p\gamma w\rangle(I)] \overset{a}{\to} C[\langle p_1\gamma_1 w\rangle(I\langle p_2\gamma_2\rangle(Nil))] \\
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Semantics

- A step modifies a thread's state according to a rule

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Execution Trees

- Binary node $\langle p\gamma \overset{a}{\hookrightarrow} p_1\gamma_1 \rhd p_2\gamma_2 \rangle(t_1, t_2)$ describes execution of spawn-step

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- Execution trees

$$XR ::= \langle Base\rangle(XR) \mid \langle Call\rangle^R(XR, XR) \mid \langle Return\rangle \mid \langle Spawn\rangle(XR, XN)$$

$$XN ::= \langle Base\rangle(XN) \mid \langle Call\rangle^N(XN) \mid \langle Call\rangle^R(XR, XN) \mid \langle P \times \Gamma\rangle \mid \langle Spawn\rangle(XN, XN)$$

List Operations

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  - The remaining equations are unchanged (Complete definition on next slide)

Reached configurations

Define $c : XN \to \mathit{conf}$ and $s : XR \to \mathit{conflist}$

$$
\begin{aligned}
c(\langle p\gamma \overset{a}{\hookrightarrow} p'\gamma'\rangle(t)) &= c(t) \\
c(\langle p\gamma \overset{\tau}{\hookrightarrow} p'\gamma_1\gamma_2\rangle^R(t_1,t_2)) &= s(t_1)c(t_2) \\
c(\langle p\gamma \overset{\tau}{\hookrightarrow} p'\gamma_1\gamma_2\rangle^N(t)) &= c(t)\gamma_2 \qquad \text{where } \langle pw\rangle\gamma(l) = \langle pw\gamma\rangle(l) \\
c(\langle p\gamma \overset{a}{\hookrightarrow} p_1\gamma_1 \rhd p_2\gamma_2\rangle(t_1,t_2)) &= [c(t_2)]c(t_1) \\
c(\langle p\gamma\rangle) &= \langle p\gamma\rangle \\
s(\langle p\gamma \overset{a}{\hookrightarrow} p'\gamma'\rangle(t)) &= s(t) \\
s(\langle p\gamma \overset{\tau}{\hookrightarrow} p'\gamma_1\gamma_2\rangle^R(t_1,t_2)) &= s(t_1)s(t_2) \\
s(\langle p\gamma \overset{a}{\hookrightarrow} p_1\gamma_1 \rhd p_2\gamma_2\rangle(t_1,t_2)) &= [c(t_2)]s(t_1) \\
s(\langle p\gamma \overset{a}{\hookrightarrow} p'\rangle) &= \mathit{Nil}
\end{aligned}
$$

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  - Complete rules on next slide

Rules for execution trees

$$
\begin{array}{rll}
p\gamma & \rightarrow \langle p\gamma \overset{a}{\hookrightarrow} p'\gamma' \rangle (p'\gamma') & \text{if } p\gamma \overset{a}{\hookrightarrow} p'\gamma' \in \Delta \\
p\gamma & \rightarrow \langle p\gamma \overset{a}{\hookrightarrow} p'\gamma_1\gamma_2 \rangle^N (p'\gamma_1) & \text{if } p\gamma \overset{a}{\hookrightarrow} p'\gamma_1\gamma_2 \in \Delta \\
p\gamma & \rightarrow \langle p\gamma \overset{a}{\hookrightarrow} p'\gamma_1\gamma_2 \rangle^R (p'\gamma_1 | p'', p''\gamma_2) & \text{if } p'' \in P \text{ and } p\gamma \overset{a}{\hookrightarrow} p'\gamma_1\gamma_2 \in \Delta \\
p\gamma & \rightarrow \langle p\gamma \overset{a}{\hookrightarrow} p_1\gamma_1 \rhd p_2\gamma_2 \rangle (p_1\gamma_1, p_2\gamma_2) & \text{if } p\gamma \overset{a}{\hookrightarrow} p_1\gamma_1 \rhd p_2\gamma_2 \in \Delta \\
p\gamma & \rightarrow \langle p\gamma \rangle & \\
p\gamma|p'' & \rightarrow \langle p\gamma \overset{a}{\hookrightarrow} p'\gamma' \rangle (p'\gamma' | p'') & \text{if } p\gamma \overset{a}{\hookrightarrow} p'\gamma' \in \Delta \\
p\gamma|p'' & \rightarrow \langle p\gamma \overset{a}{\hookrightarrow} p'\gamma_1\gamma_2 \rangle^R (p'\gamma_1 | p''', p'''\gamma_2 | p'') & \text{if } p''' \in P \text{ and } p\gamma \overset{a}{\hookrightarrow} p'\gamma_1\gamma_2 \in \Delta \\
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p\gamma|p'' & \rightarrow \langle p\gamma \overset{\tau}{\hookrightarrow} p'' \rangle & \text{if } p\gamma \overset{\tau}{\hookrightarrow} p'' \in \Delta
\end{array}
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Relating Execution Trees and DPN Semantics

Theorem

Let M be a DPN. Then $\exists l.\ p_0\gamma_0 \xrightarrow{l}{}^{*} c'$ iff $\exists t.\ t \in L(\mathcal{A}_M) \wedge c(t) = c'$

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  - They are *interleavings* of the thread's action sequences
  - One execution tree corresponds to many such interleavings

Interleaving

- We define $s_1 \otimes s_2$ to be the set of *interleavings* of lists s_1 and s_2

$$s_1 \otimes \varepsilon = \{s_1\} \qquad \varepsilon \otimes s_2 = \{s_2\}$$
$$a_1 s_1 \otimes a_2 s_2 = a_1(s_1 \otimes a_2 s_2) \cup a_2(a_1 s_1 \otimes s_2)$$

- Intuitively: All sequences of steps that may be observed if one thread executes s_1 and another independently executes s_2.

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$$\langle pw\rangle([c]) \xrightarrow{s}{}^{*} \langle p'w'\rangle(l') \iff$$
$$\exists c'\ l''\ s_1\ s_2.\ l' = c'l'' \wedge s \in s_1 \otimes s_2 \wedge \langle pw\rangle(\varepsilon) \xrightarrow{s_1}{}^{*} \langle p'w'\rangle(l'') \wedge c \xrightarrow{s_2}{}^{*} c'$$

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- Proof, by induction on number of steps:

$$\langle p\gamma\rangle(\varepsilon) \rightarrow^{*} \langle p'\rangle(c') \iff \exists t.p\gamma|p' \rightarrow t \wedge s(t) = c'$$
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  - And, by definition of $s()$, we have

$$s(\langle r\rangle(t_1, t_2)) = [c(t_2)]s(t_1) = c'l'' = l' \quad \square$$

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  - If the analysis returns yes, such a configuration may be reachable

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  - If the analysis returns no, we are sure that no such configuration is reachable
  - If the analysis returns yes, such a configuration may be reachable
    - Or it may be a false positive due to over-approximation

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- Used to protect access to shared resources
- We assume there is a finite set $\mathbb{L}$ of locks, and the actions $[_l$ (acquire) and $]_l$ (release) for every $l \in \mathbb{L}$

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  - Emptiness of intersection of CF-Languages
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  - Bind locks to procedures: Acquisition on call, release on return

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  - Idea: Run two PDS concurrently, and ensure that sequences of base transitions must run in lock-step
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  - Check for simultaneous reachability of final states

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Thread 1:			$[_{0?}$	$]_{0!}$			$[_{0}$	$]_{0?}$			$[_{0!}$	$]_{0}$
Thread 2:	$[_{0}$	$]_{0?}$			$[_{0!}$	$]_{0}$			$[_{0?}$	$]_{0!}$		

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$$\begin{array}{l|cccccccccccc}
\text{Thread 1:} & & & [_{0?} &]_{0!} & & & [_{0} &]_{0?} & & & [_{0!} &]_{0} \\
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\end{array}$$

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- The sequences for producing 1 are analogously

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  - If one thread starts before the other has finished initialization, the other will be stuck at $[_{l_i}]_{l_i}$ forever
- Thus, final states of PDSs simultaneously reachable, iff encoded CF-languages have non-empty intersection

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  - Enforce correct order of guessing assignment and checking: One additional lock

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    - i.e. whether there is a valuation of the variables such that Φ holds.

Reduction to 3-SAT

```
ass(i):
  if ... then {
    acquire x_i  ass(i+1)  release x_i
  } else {
    acquire x̄_i  ass(i+1)  release x̄_i
  }
  return

ass(n+1):
  acquire(s); release(s);
  label1: return

thread1: ass(1)
```

```
check(i):
  if (...) {
    acquire l_i1 ; release l_i1 ;
  } else if (...) {
    acquire l_i2 ; release l_i2 ;
  } else {
    acquire l_i3 ; release l_i3 ;
  }

thread2:
  acquire(s);
  check(1); ...; check(m);
  label2: skip
  release(s)
```

- `label1` and `label2` simultaneously reachable, iff formula is satisfiable.

Last Lecture

- Execution trees of DPN
- Locks: Negative results
  - Reachability in DPN (even 2-PDS) wrt. arbitrary locking is undecidable
    - Reduction to deciding intersection of CF languages
  - Reachability in DPN (even 2-PDS) wrt. nested locking is NP-hard
    - Reduction to 3-SAT

Table of Contents

2-PDS with locks

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$$(p\gamma w_1, p_2 w_2, L) \xrightarrow{a}_{\mathrm{ls}} (p' w' w_1, p_2 w_2, eff(a, L)) \quad \text{if } p\gamma \overset{a}{\hookrightarrow} p' w' \in \Delta \text{ and } cond(a, L) \qquad \text{(left)}$$

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&\qquad \text{iff } \exists l_1, l_2.\ p_1 w_1 \xrightarrow{l_1}{}^{*} p_1' w_1' \wedge p_2 w_2 \xrightarrow{l_2}{}^{*} p_2' w_2' \wedge (l_1, l_2, L) \xrightarrow{l}{}^{*} (\varepsilon, \varepsilon, L')
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  - Proof: Straightforward simulation proof

Execution trees of 2-PDS

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  - Where $c : XN \to \mathit{conf}$ extracts reached configuration from execution tree and $a : XN \to \mathrm{Act}^*$ extracts labeling sequence from execution tree (cf. Homework 9.2)

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- Thus, we get a tree automaton for schedulable execution trees.
- Checking the intersection of this, the tree automaton for execution trees, and the error property for emptiness gives us lock-sensitive model-checker

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- Lemma

$$
(l_1, l_2, \emptyset) \rightarrow^* (\varepsilon, \varepsilon, \_) \text{ iff } A(l_1) \cap A(l_2) = \emptyset \wedge \mathrm{acyclic}(G(l_1) \cup G(l_2))
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    - Schedule usages of locks first
    - If both, l_1 and l_2 start with final acquisitions:
      Choose acquisition that comes first in topological ordering of $G(l_1) \cup G(l_2)$

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- In practice: Avoid computing unnecessary states of tree automata

Last Lecture

- 2-PDS with locks
- Acquisition histories
- Deciding lock-sensitive reachability

Table of Contents

DPNs with locks

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- Step relation:

$$(c, L) \xrightarrow{a} (c', \mathit{eff}(a, L)) \text{ iff } \mathit{cond}(a, L) \wedge c \xrightarrow{a} c'$$

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- Terminated scheduling tree: All steps are executed, i.e., all nodes labeled with *Nil*

$$ST_{term} ::= \langle Nil \rangle (SL_{term}) \qquad SL_{term} ::= Nil \mid Cons(ST_{term}, SL_{term})$$

Operations on Branching Lists

- Generalized concatenation

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\begin{aligned}
(\mathit{Nil})l' &:= l' \\
\mathit{Cons}(a, l)l' &:= \mathit{Cons}(a, ll') \\
\mathit{Spawn}(a, l_1, l_2)l' &:= \mathit{Spawn}(a, l_1 l', l_2)
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- This thread's steps: $this : BL \to \mathrm{Act}^*$

$$\begin{aligned}this(Nil) &:= Nil\\ this(Cons(a,l)) &:= Cons(a,this(l))\\ this(Spawn(a,l_1,l_2)) &= Cons(a,this(l_1))\end{aligned}$$

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- Set of steps

$$
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x \in \mathit{Nil} &:= \mathit{false} \\
x \in \mathit{Cons}(a, l) &:= x = a \vee x \in l \\
x \in \mathit{Spawn}(a, l_1, l_2) &:= x = a \vee x \in l_1 \vee x \in l_2
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Relation of execution tree and scheduling tree

- Execution trees correspond to scheduling trees: $st : XN \rightarrow ST$ and $st' : XN \rightarrow BL$ where

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st(t) &:= \langle st'(t) \rangle (Nil) \\
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st'(\langle p\gamma \overset{a}{\hookrightarrow} p_1\gamma_1 \rhd p_2\gamma_2 \rangle(t_1, t_2)) &:= Spawn(a, st'(t_1), st'(t_2)) \\
st'(\langle p\gamma \overset{a}{\hookrightarrow} p'\gamma_1\gamma_2 \rangle^N(t)) &:= Cons(a, st'(t)) \\
st'(\langle p\gamma \overset{a}{\hookrightarrow} p'\gamma_1\gamma_2 \rangle^R(t_1, t_2)) &:= [a] st'(t_1) st'(t_2) \\
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- It can be proved

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\begin{aligned}
&(\langle p_0\gamma_0\rangle(\varepsilon), \emptyset) \xrightarrow{l}{}^{*} (c', L) \\
&\iff \exists t \in XN.\, \exists t' \in ST_{term}.\ t \in L(\mathcal{A}_M) \wedge c(t) = c' \wedge (st(t), \emptyset) \xrightarrow{l}{}^{*} (t', L)
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Relation of execution tree and scheduling tree

- Execution trees correspond to scheduling trees: $st : XN \to ST$ and $st' : XN \to BL$ where

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  - Note: This proof requires a generalization from a single-thread start configuration to arbitrary start configurations.

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- Compute set of final acquisitions

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A(Cons(a, l)) &= A(l) && \text{if } a \in \mathrm{Act}_{nl} \text{ or } a =]_x \text{ for } x \in \mathbb{L} \\
A(Cons([_x, l)) &= A(l) && \text{if }]_x \in this(l) \\
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G(Cons([_x, l)) &= G(l) && \text{if }]_x \in this(l) \\
G(Cons([_x, l)) &= G(l) \cup \{x\} \times \mathrm{acq}(l) && \text{if }]_x \notin this(l)
\end{aligned}
$$

where $\mathrm{acq}(l) := \{x \mid [_x \in l\}$

Acquisition Graphs characterize Schedulability

- For scheduling tree $\langle bl \rangle(Nil) \in ST$ and labeling sequence $l \in \mathrm{Act}^*$, we have

$$\exists t'.(\langle bl \rangle(Nil), \emptyset) \xrightarrow{l}{}^{*} (t', L) \wedge t' \in ST_{term} \iff \mathrm{acyclic}(G(bl)) \wedge fac(bl)$$

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    - Formally: Generalize to initial set of locks disjoint from locks that occur in scheduling tree. Generalize to arbitrary scheduling tree. Induction on scheduling tree.

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- Schedulable scheduling trees are regular (compute acquisition graphs by tree automata)
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- Thus, we can decide lock-sensitive reachability of a regular set of configurations of a DPN.

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    - Set of used locks only required at final acquisition.
    - Just check that less locks are used afterwards
    - Accepts executions with the guess acquisition graph, or with smaller ones

Main Theorem

Lock-sensitive reachability of a regular set of configurations is NP-complete for DPNs

Complexity of related problems

	DPN	PPDS	2PDS	DFN	PFSM	*n*FSM
$\mathrm{EF}(p_1 \parallel p_2)$	$\mathrm{NP}^{*?}$	$\mathrm{NP}^{\dagger?}$	$\underline{\mathrm{NP}}^{\dagger?}$	$\underline{\mathrm{NP}}^{*!}$	P	P
$\mathrm{EF}(A)$	NP	NP	$\mathrm{NP}^{\dagger?}$	NP	$\underline{\mathrm{NP}}$	P
$\mathrm{EF}(p_1 \parallel p_2 \wedge \mathrm{EF}(p_3 \parallel p_4))$	NP	NP	$\underline{\mathrm{NP}}$	$\mathrm{NP}^{*!}$	P	P
$\mathrm{EF}(A_1 \wedge \mathrm{EF}(A_2))$	NP	NP	NP	NP	NP	P
$\mathrm{EF}^{\setminus \text{neg}}$ (fixed #ops)	NP	NP	NP	NP	NP	P
EF (fixed #ops)	$\underline{\geq}$ $\underline{\mathrm{PSPACE}}^{\ddagger}$			$\underline{\geq}$NP		P
$\mathrm{EF}^{\setminus \text{neg}}$	$\underline{\geq}$ $\underline{\mathrm{PSPACE}}^{\ddagger \text{reg?}}$				$\underline{\geq}$ $\underline{\mathrm{NP}}^{\ddagger}$	P
EF	$\underline{\geq}$ $\underline{\mathrm{PSPACE}}^{\ddagger}$					P

- $*$ Requires spawn inside lock
  - $*!$ Polynomial algorithm if no spawn inside lock
  - $*?$ Complexity unknown if no spawn inside lock
- $\dagger?$ Hardness proof requires deadlocks/escapable locks. Complexity without this unknown.
- $\ddagger$ Hardness result requires no locks
- reg? Hardness requires regular APs. Complexity for double-indexed APs unknown ($\geq$NP)

The End

Thank you for listening