Program Optimization

Peter Lammich

WS 2016/17

Overview by Lecture

Organizational Issues

Lectures Wed 10:15-11:45 and Thu 10:15-11:45 in MI 00.13.009A

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Exam	Written (or Oral), Bonus for Homework!
	• $\geq$ 50% of homework $\Longrightarrow$ 0.3/0.4 better grade On first exam attempt. Only if passed w/o bonus!

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How many of you are attending “Semantics” lecture?

Info-2 Tutors

We need tutors for Info II lecture. If you are interested, please contact Julian Kranz `julian.kranz@in.tum.de`.

Proposed Content

- Avoiding redundant computations
  - E.g. Available expressions, constant propagation, code motion
- Replacing expensive with cheaper computations
  - E.g. peep hole optimization, inlining, strength reduction
- Exploiting Hardware
  - E.g. instruction selection, register allocation, scheduling
- Analysis of parallel programs
  - E.g. threads, locks, data-races

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Observation 1

Intuitive programs are often inefficient

```
void swap (int i, int j) {
  int t;
  if (a[i] > a[j]) {
    t = a[j];
    a[j] = a[i];
    a[i] = t;
  }
}
```

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  - Values loaded 2 times

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    a[i] = t;
  }
}
```

- Inefficiencies
  - Addresses computed 3 times
  - Values loaded 2 times
- Improvements
  - Use pointers for array indexing
  - Store the values of `a[i]`, `a[j]`

```
void swap (int *p, int *q) {
  int t, ai, aj;
  ai=*p; aj=*q;
  if (ai > aj) {
    t = aj;
    *q = ai;
    *p = t; // t can also be eliminated
  }
}
```

```c
void swap (int *p, int *q) {
  int ai, aj;
  ai=*p; aj=*q;
  if (ai > aj) {
    *q = ai;
    *p = aj;
  }
}
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void swap (int *p, int *q) {
  int ai, aj;
  ai=*p; aj=*q;
  if (ai > aj) {
    *q = ai;
    *p = aj;
  }
}
```

Caveat: Program less intuitive

Observation 2

High-level languages (even C) abstract from hardware (and efficiency)

Compiler needs to transform intuitively written programs to hardware.

Examples

- Filling of delay slots
- Utilization of special instructions
- Re-organization of memory accesses for better cache behavior
- Removal of (useless) overflow/range checks

Observation 3

Program improvements need not always be correct

- E.g. transform `f() + f()` to `2*f()`

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Program improvements need not always be correct

- E.g. transform `f() + f()` to `2*f()`
- Idea: Save second evaluation of `f`
- But what if `f` has side-effects or reads input?

Insight

- Program optimizations have preconditions
- These must be
  - Formalized
  - Checked
- It must be proved that optimization is correct
  - I.e., preserves semantics

Observation 4

Optimizations techniques depend on programming language

- What inefficiencies occur
- How analyzable is the language
- How difficult it is to prove correctness

Example: Java

- (Unavoidable) inefficiencies
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Example: Java

- (Unavoidable) inefficiencies
  - Array bound checks
  - Dynamic method invocation
  - Bombastic object organization
- Analyzability
  - + No pointer arithmetic, no pointers into stack
  - - Dynamic class loading
  - - Reflection, exceptions, threads
- Correctness proof
  - + Well-defined semantics (more or less)
  - - Features, features, features
  - - Libraries with changing behavior

In this course

- Simple imperative programming language

`R = e`	Assignment
`R = M[e]`	Load
`M[e₁] = e₂` → $M[e_1] = e_2$	Store
`if` `(e) ...` **`else`** `...`	Conditional branching
`goto` `label`	Unconditional branching

R Registers, assuming infinite supply

e Integer-valued expressions over constants, registers, operators

M Memory, addressed by integer $\geq$ 0, assuming infinite memory

Note

- For the beginning, we omit procedures
  - Focus on intra-procedural optimizations
  - External procedures taken into account via statement `f()`
    - unknown procedure
    - may arbitrarily mess around with memory and registers
- Intermediate Language, in which (almost) everything can be translated

Example: Swap

```
void swap (int i, int j) {
  int t;
  if (a[i] > a[j]) {
    t = a[j];
    a[j] = a[i];
    a[i] = t;
  }
}
```

```
A1 = A0 + 1*i   //R1 = a[i]
R1 = M[A1]
A2 = A0 + 1*j   //R2 = a[j]
R2 = M[A2]
if (R1 > R2) {
  A3 = A0 + 1*j //t=a[j]
  t = M[A3]
  A4 = A0 + 1*j //a[j] = a[i]
  A5 = A0 + 1*i
  R3 = M[A5]
  M[A4] = R3
  A6 = A0 + 1*i //a[i]=t
  M[A6] = t
}
```

Example: Swap

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```

Assume A_0 contains address of array a

1: $A_1 = A_0 + 1*i$ *//R_1 = a[i]*
2: $R_1 = M[A_1]$
3: $A_2 = A_0 + 1*j$ *//R_2 = a[j]*
4: $R_2 = M[A_2]$
5: **if** $(R_1 > R_2)$ {
6:    $A_3 = A_0 + 1*j$ *//t=a[j]*
7:    $t = M[A_3]$
8:    $A_4 = A_0 + 1*j$ *//a[j] = a[i]*
9:    $A_5 = A_0 + 1*i$
0:    $R_3 = M[A_5]$
1:    $M[A_4] = R_3$
2:    $A_6 = A_0 + 1*i$ *//a[i]=t*
3:    $M[A_6] = t$
}

Optimizations

1. $\texttt{1 * R} \mapsto \texttt{R}$
2. Re-use of sub-expressions

$$\texttt{A}_1 \texttt{ == } \texttt{A}_5\texttt{== } \texttt{A}_6\texttt{, }\texttt{A}_2 \texttt{ == } \texttt{A}_3\texttt{== } \texttt{A}_4$$
$$\texttt{M[A}_1\texttt{] == M[A}_5\texttt{],M[A}_2\texttt{] == M[A}_3\texttt{]}$$
$$\texttt{R}_1 \texttt{ == } \texttt{R}_3$$
$$\texttt{R}_2 \texttt{ = t}$$

Now we have

```
A1 = A0 + i
R1 = M[A1]
A2 = A0 + j
R2 = M[A2]
if (R1 > R2) {
  M[A2] = R1
  M[A1] = R2
}
```

Original was:

```
A1 = A0 + 1*i   //R1 = a[i]
R1 = M[A1]
A2 = A0 + 1*j   //R2 = a[j]
R2 = M[A2]
if (R1 > R2) {
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  t = M[A3]
  A4 = A0 + 1*j //a[j] = a[i]
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  M[A4] = R3
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}
```

Gain

	before	after
$+$	6	2
$*$	6	0
$>$	1	1
load	4	2
store	2	2
$R =$	6	2

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Idea

If same value is computed repeatedly

- Store it after first computation
- Replace further computations by look-up

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Method

- Identify repeated computations
- Memorize results
- Replace re-computation by memorized value

Example

```
    x = 1
    y = M[42]
A:  r1 = x + y
    ...
B:  r2 = x + y
```

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     x  =  1
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- Repeated computation of `x+y` at `B`, if
  - `A` is always executed before `B`
  - `x+y` has the same value at *A* and *B*.

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    x = 1
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```

- Repeated computation of `x+y` at `B`, if
  - `A` is always executed before `B`
  - `x+y` has the same value at *A* and *B*.
- We need
  - Operational semantics
  - Method to identify (at least some) repeated computations

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Rice's theorem (informal)

All non-trivial semantic properties of a Turing-complete programming language are undecidable.

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But Still can use approximate approaches

- Approximation of semantic property
- Show that transformation is still correct

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But Still can use approximate approaches

- Approximation of semantic property
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Example: Only identify subset of repeated computations.

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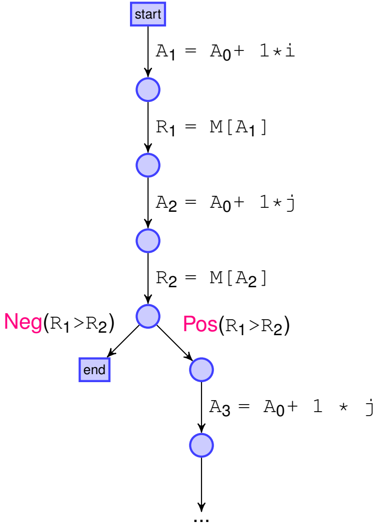
Small-step operational semantics

Intuition: Instructions modify state (registers, memory)

Represent program as control flow graph (CFG)



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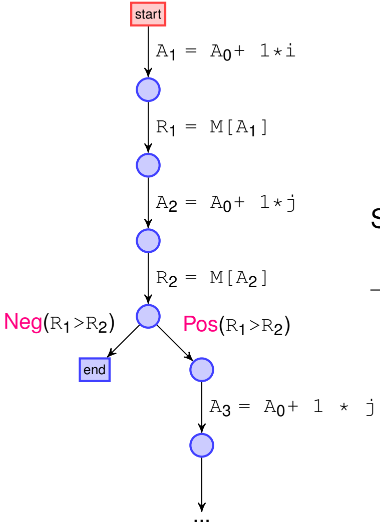

State:

A_0	M[0..4]	i	j	A_1	A_2	R_1	R_2
0	1,2,3,4,5	2	4	-	-	-	-

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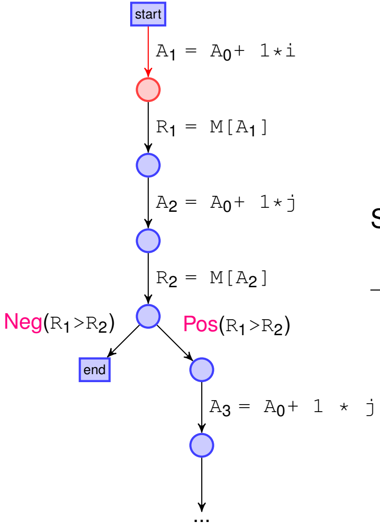

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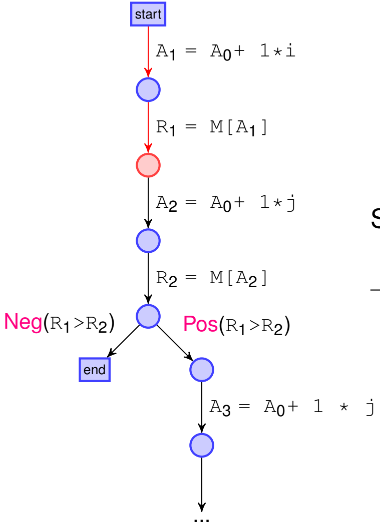

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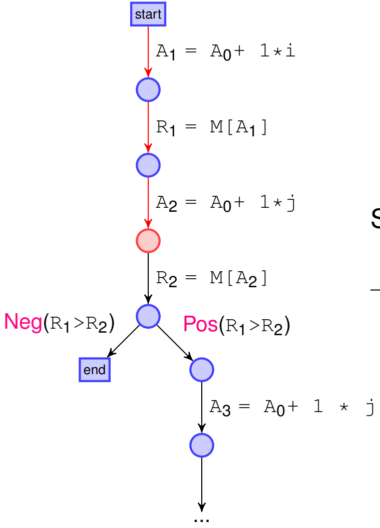

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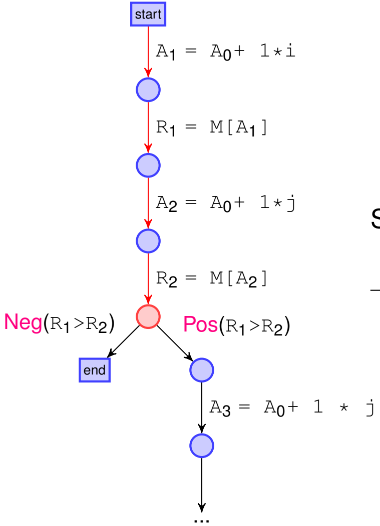

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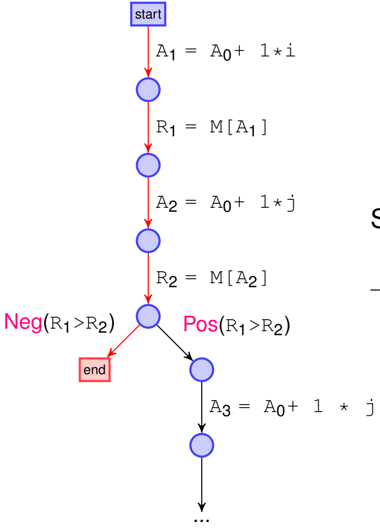

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Definition (Registers and Expressions)

Reg is an infinite set of register names. Expr is the set of expressions over these registers, constants and a standard set of operations.

Note: We do not formally define the set of operations here

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$\mathrm{Act} = \mathrm{Nop} \mid \mathrm{Pos}(e) \mid \mathrm{Neg}(e) \mid R = e \mid R = M[e] \mid M[e_1] = e_2$
where $e, e_1, e_2 \in \mathrm{Expr}$ are expressions and $R \in \mathrm{Reg}$ is a register.

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Definition (Control Flow Graph)

An edge-labeled graph $G = (V, E, v_0, V_{\mathrm{end}})$ where $E \subseteq V \times \mathrm{Act} \times V$, $v_0 \in V$, $V_{\mathrm{end}} \subseteq V$ is called control flow graph (CFG).

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Definition (State)

A state $s \in \mathrm{State}$ is represented by a pair $s = (\rho, \mu)$, where

ρ : $\mathrm{Reg} \to \mathrm{int}$ is the content of registers

μ : $\mathrm{int} \to \mathrm{int}$ is the content of memory

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[\![R = e]\!](\rho,\mu) &:= (\rho(R \mapsto [\![e]\!]\rho),\mu) \\
[\![R = M[e]]\!](\rho,\mu) &:= (\rho(R \mapsto \mu([\![e]\!]\rho)),\mu) \\
[\![M[e_1] = e_2]\!](\rho,\mu) &:= (\rho,\mu([\![e_1]\!]\rho \mapsto [\![e_2]\!]\rho))
\end{aligned}
$$

Formally (III)

Given a CFG $G = (V, E, v_0, V_{\mathrm{end}})$

Definition (Path)

A sequence of adjacent edges $\pi = (v_1, a_1, v_2)(v_2, a_2, v_3) \ldots (v_n, a_n, v_{n+1}) \in E^*$ is called path from v_1 to v_{n+1}.

Notation $v_1 \xrightarrow{\pi} v_{n+1}$

Convention π is called path to v iff $v_0 \xrightarrow{\pi} v$

Special case $v \xrightarrow{\varepsilon} v$ for any $v \in V$

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Definition (Effect of edge and path)

The effect of an edge $k = (u, a, v)$ is the effect of its action:

$$[\![(u, a, v)]\!] := [\![a]\!]$$

The effect of a path $\pi = k_1 \ldots k_n$ is the composition of the edge effects:

$$[\![k_1 \ldots k_n]\!] := [\![k_n]\!] \circ \ldots \circ [\![k_1]\!]$$

Formally (IV)

Definition (Computation)

A path π is called computation for state s, iff its effect is defined on s, i.e.,

$$s \in \mathrm{dom}(\llbracket\pi\rrbracket)$$

Then, the state $s' = \llbracket\pi\rrbracket s$ is called result of the computation.

Summary

- Action: $\mathrm{Act} = \mathrm{Nop} \mid \mathrm{Pos}(e) \mid \mathrm{Neg}(e) \mid R = e \mid R = M[e] \mid M[e_1] = e_2$
- CFG: $G = (V, E, v_0, V_{\mathrm{end}})$, $E \subseteq V \times \mathrm{Act} \times V$
- State: $s = (\rho, \mu)$, $\rho : \mathrm{Reg} \to \mathrm{int}$ (registers), $\mu : \mathrm{int} \to \mathrm{int}$ (memory)
- Value of expression under ρ: $[\![e]\!]\rho : \mathrm{int}$
- Effect of action a: $[\![a]\!] : \mathrm{State} \to \mathrm{State}$ (partial)
- Path π: Sequence of adjacent edges
- Effect of edge $k = (u, a, v)$: $[\![k]\!] = [\![a]\!]$
- Effect of path $\pi = k_1 \dots k_n$: $[\![\pi]\!] = [\![k_n]\!] \circ \dots \circ [\![k_1]\!]$
- π is computation for s: $s \in \mathrm{dom}([\![\pi]\!])$
- Result of computation π for s: $[\![\pi]\!]s$

Table of Contents

Memorization

First, let's memorize every expression

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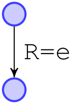

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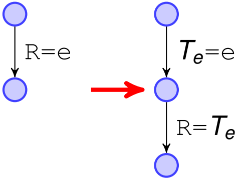

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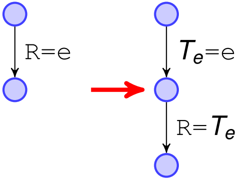

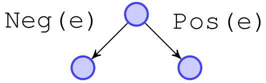

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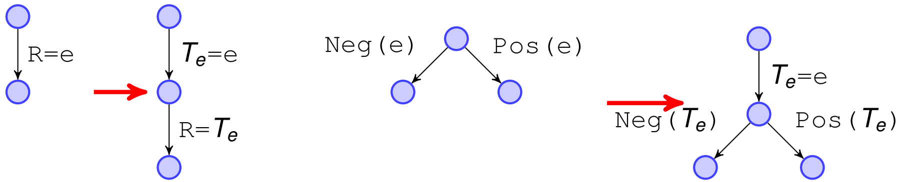

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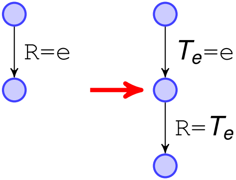

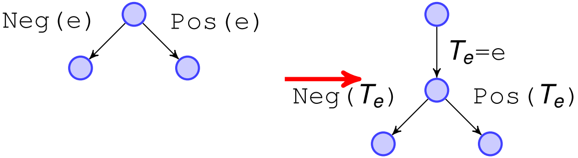

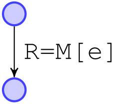

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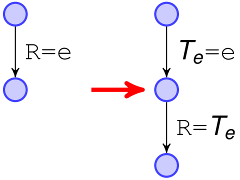

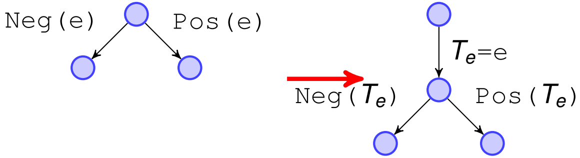

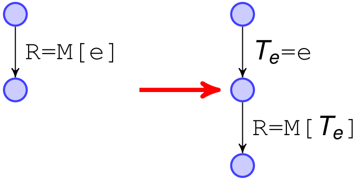

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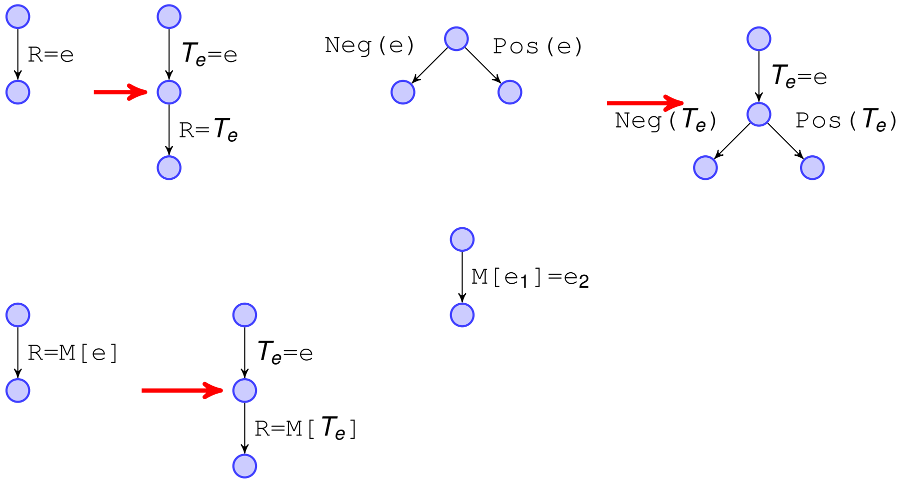

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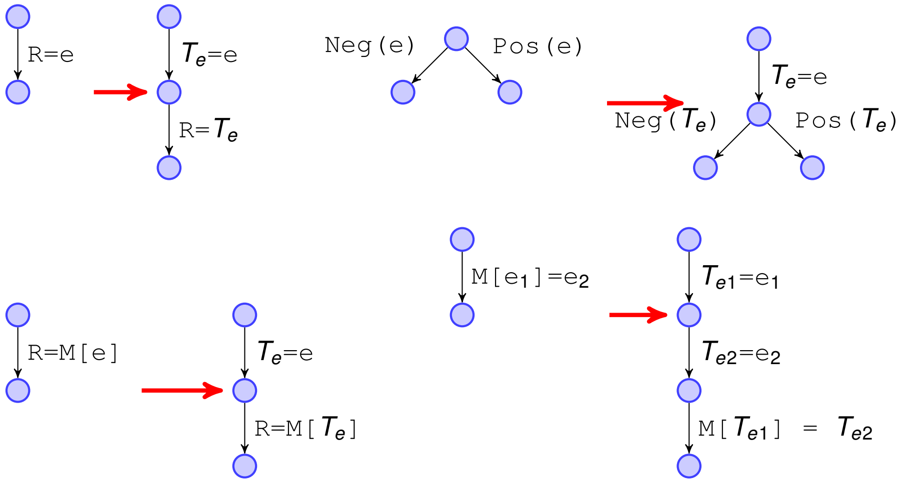

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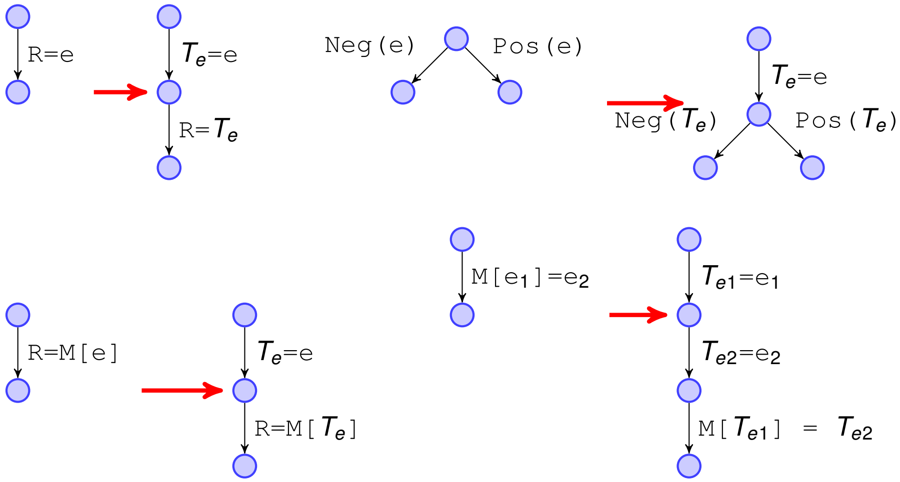

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Last Lecture (Oct 20)

- Simple intermediate language (IL)
  - Registers, memory, cond/ucond branching
  - Compiler: Input $\rightarrow$ Intermediate Language $\rightarrow$ Machine Code
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- Memorization Transformation
  - Memorize evaluation of e in register T_e

Available Expressions (Semantically)

Definition (Available Expressions in state)

The set of semantically available expressions in state (ρ, μ) is defined as

$$\mathrm{Aexp}(\rho, \mu) := \{e \mid [\![e]\!]\rho = \rho(T_e)\}$$

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Border case: All expressions available in undefined state

$$\mathrm{Aexp}(\text{undefined}) := \mathrm{Expr}$$

(See next slide why this makes sense)

Available Expressions (Semantically)

Definition (Available Expression at program point)

The set $\mathrm{Aexp}(u)$ of semantically available expressions at program point u is the set of expressions that are available in all states that may occur when the program is at u.

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Note Above definition is smoother due to $\mathrm{Aexp}(\text{undefined}) := \mathrm{Expr}$

Simple Redundancy Elimination

Transformation Replace edge $(u, T_e = e, v)$ by (u, Nop, v) if e semantically available at u.

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Remaining Problem: How to compute available expressions

Precisely: No chance (Rice's Theorem)

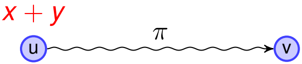
Observation: Enough to compute subset of semantically available expressions

- Transformation still correct

Available Expressions (Syntactically)

Idea Expression e (syntactically) available after computation π

- if e has been evaluated, and no register of e has been assigned afterwards



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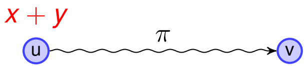

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Available Expressions (Syntactically)

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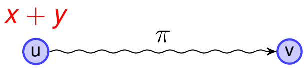

π *does not contain assignment to x nor y*

Purely syntactic criterion

Can be computed incrementally for every edge

Available Expressions (Computation)

Let A be a set of available expressions.

Recall: Available $\Longleftarrow$ Already evaluated and no reg. assigned afterwards

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An action a transforms this into the set $[\![a]\!]^{\#} A$ of expressions available after a has been executed

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[\![R = T_e]\!]^{\#} A &:= A \setminus \mathrm{Expr}_R \qquad \mathrm{Expr}_R := \text{expressions containing } R \\
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Definition (Available at v)

The set $A[v]$ of (syntactically) available expressions at v is

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Available Expressions (Correctness)

Idea Abstract effect corresponds to concrete effect

Lemma

$A \subseteq \mathit{Aexp}(s) \implies [\![a]\!]^{\#} A \subseteq \mathit{Aexp}([\![a]\!] s)$

Proof Check for every type of action.

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And to program points

$$A[u] \subseteq \mathrm{Aexp}(u)$$

Recall:

$$\mathrm{Aexp}(u) = \bigcap\{\mathrm{Aexp}([\![\pi]\!]s) \mid \pi, s.\ v_0 \xrightarrow{\pi} u\}$$
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Summary

1. Transform program to memorize everything
   - Introduce registers T_e
2. Compute $A[u]$ for every program point u
   - $A[u] = \bigcap\{[\![\pi]\!]^\# \emptyset \mid \pi.\ v_0 \xrightarrow{\pi} u\}$
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Warning Memorization transformation for $R = e$ should only be applied if

- $R \notin \mathrm{Reg}(e)$ (Otherwise, expression immediately unavailable)
- $e \notin \mathrm{Reg}$ (Otherwise, only one more register introduced)
- Evaluation of e is nontrivial (Otherwise, re-evaluation cheaper than memorization)

Remaining Problem

How to compute $A[u] = \bigcap\{[\![\pi]\!]^{\#}\emptyset \mid v_0 \xrightarrow{\pi} u\}$

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Intuition

Nothing available at start node
For edge (u, a, v): At v, at most those expressions are available that would be available if we come from u.

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Let's regard a slightly modified available expression analysis

- Available expressions before memorization transformation has been applied
- Yields smaller examples, but more complicated proofs :)

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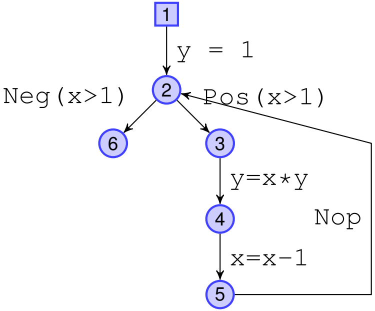
Effect of transformation already included in constraint system

Example



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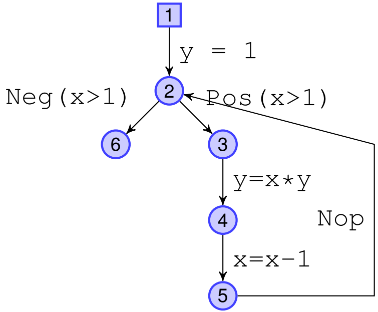

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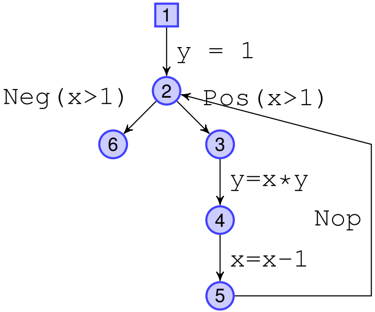

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$$A[2] \subseteq A[5]$$

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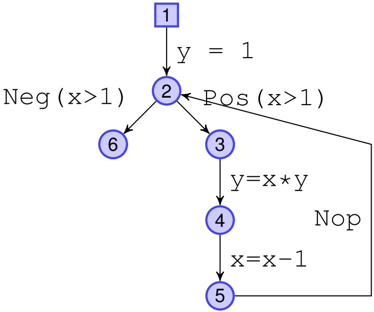

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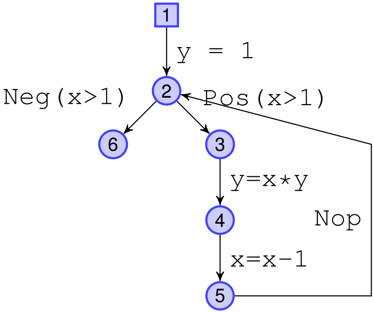

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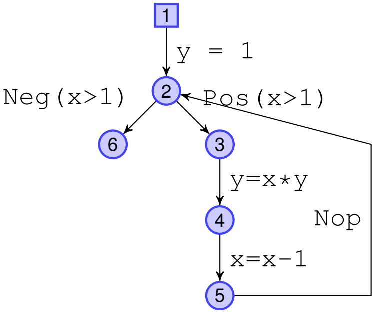

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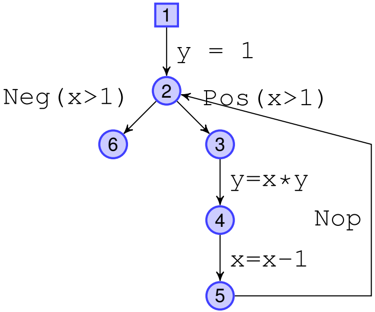

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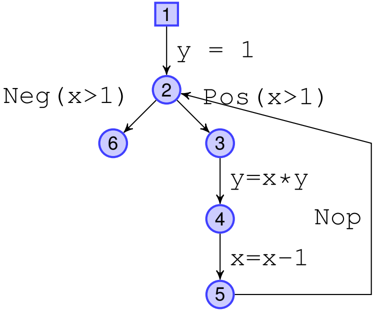

Solution:

$$A[1] = \emptyset$$
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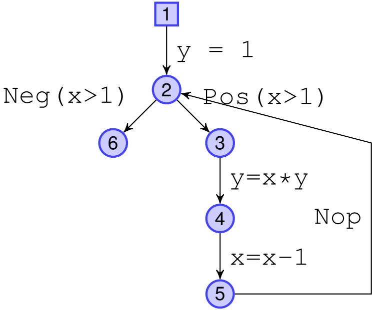
Example



Also a solution:

$$A[1] = \emptyset$$
$$A[2] = \emptyset$$
$$A[3] = \emptyset$$
$$A[4] = \emptyset$$
$$A[5] = \emptyset$$
$$A[6] = \emptyset$$

Wanted

- Maximally large solution
  - Intuitively: Most precise information

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- Maximally large solution
  - Intuitively: Most precise information
- An algorithm to compute this solution

Naive Fixpoint Iteration (Sketch)

1. Initialize every $A[u] = \mathrm{Expr}$
   - Expressions actually occurring in program!
2. Evaluate RHSs
3. Update LHSs by intersecting with values of RHSs
4. Repeat (goto 2) until values of $A[u]$ stabilize

Naive Fixpoint Iteration (Example)

- On whiteboard!

Naive Fixpoint Iteration (Correctness)

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Why does it compute the maximal solution?

- Fixed-point theory. (Comes next)

Table of Contents

Partial Orders

Definition (Partial Order)

A partial order $(\mathbb{D}, \sqsubseteq)$ is a relation $\sqsubseteq$ on $\mathbb{D}$ that is reflexive, antisymmetric, and transitive, i.e., for all $a, b, c \in \mathbb{D}$:

$$\begin{aligned} a &\sqsubseteq a && \text{(reflexive)} \\ a \sqsubseteq b \wedge b \sqsubseteq a \implies a &= b && \text{(antisymmetric)} \\ a \sqsubseteq b \sqsubseteq c \implies a &\sqsubseteq c && \text{(transitive)} \end{aligned}$$

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Examples $\leq$ on $\mathbb{N}$, $\subseteq$. Also $\geq$, $\supseteq$

Lemma (Dual order)

We define $a \sqsupseteq b := b \sqsubseteq a$. Let $\sqsubseteq$ be a partial order on $\mathbb{D}$. Then $\sqsupseteq$ also is a partial order on $\mathbb{D}$.

More examples

$\mathbb{D} = 2^{\{a,b,c\}}$ with $\subseteq$

$$
\begin{tikzcd}[arrows=dash]
 & \{a,b,c\} \arrow[dl] \arrow[d] \arrow[dr] & \\
\{a,b\} \arrow[d] \arrow[dr] & \{a,c\} \arrow[dl] \arrow[dr] & \{b,c\} \arrow[dl] \arrow[d] \\
\{a\} \arrow[dr] & \{b\} \arrow[d] & \{c\} \arrow[dl] \\
 & \emptyset &
\end{tikzcd}
$$

More examples

$\mathbb{Z}$ with relation $=$

$$\cdots \qquad -2 \qquad -1 \qquad 0 \qquad 1 \qquad 2 \qquad \cdots$$

More examples

$\mathbb{Z}$ with relation $\leq$

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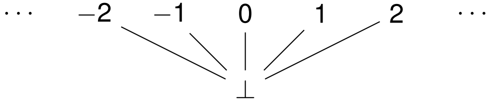
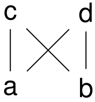
More examples

$\mathbb{Z}_\perp := \mathbb{Z} \cup \{\perp\}$ with relation $x \sqsubseteq y$ iff $x = \perp \vee x = y$



More examples

$\{a, b, c, d\}$ with $a \sqsubset c, a \sqsubset d, b \sqsubset c, b \sqsubset d$



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Upper bound not always exists, e.g. $\{0, 2, 4, \ldots\} \subseteq \mathbb{Z}$

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Upper bound not always exists, e.g. $\{0, 2, 4, \ldots\} \subseteq \mathbb{Z}$
Least upper bound not always exists, e.g. $\{a, b\} \subseteq \{a, b, c, d\}$ with $a \sqsubset c, a \sqsubset d, b \sqsubset c, b \sqsubset d$

Complete Lattice

Definition (Complete Lattice)

A complete lattice $(\mathbb{D}, \sqsubseteq)$ is a partial order where every subset $X \subseteq \mathbb{D}$ has a least upper bound $\bigsqcup X \in \mathbb{D}$.

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Moreover $a \sqcup b := \bigsqcup\{a, b\}$ and $a \sqcap b := \bigsqcap\{a, b\}$

Examples

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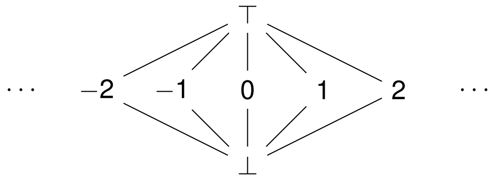
- $(\mathbb{Z}, =)$ is not. Nor is $(\mathbb{Z}, \leq)$
- $(\mathbb{Z}_\perp, \sqsubseteq)$ is also no complete lattice
  - But we can define flat complete lattice

Flat complete lattice over $\mathbb{Z}$

$\mathbb{Z}_\perp^\top := \mathbb{Z} \cup \{\perp, \top\}$ with relation $x \sqsubseteq y$ iff $x = \perp \vee y = \top \vee x = y$



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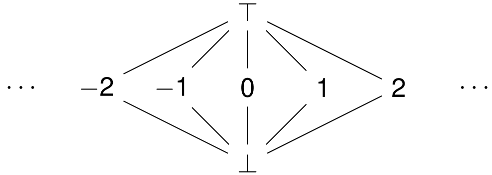

Note This construction works for every set, not only for $\mathbb{Z}$.

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Theorem

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  - Obvious: $\bigsqcup L$ is $\sqsupseteq$ than all lower bounds

Examples

- In $(2^{\{a,b,c\}}, \subseteq)$
  - Note, in lattices with $\subseteq$-ordering, we occasionally write $\bigcup$, $\bigcap$ instead of $\bigsqcup$, $\bigsqcap$
  - $\bigcup\{\{a,b\},\{a,c\}\} = \{a,b,c\}$, $\bigcap\{\{a,b\},\{a,c\}\} = \{a\}$

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- In $\mathbb{Z}_{-\infty}^{+\infty}$:
  - $\bigsqcup\{1,2,3,4\} = 4$, $\bigsqcap\{1,2,3,4\} = 1$
  - $\bigsqcup\{1,2,3,4,\ldots\} = +\infty$, $\bigsqcap\{1,2,3,4,\ldots\} = 1$

Last Lecture

- Syntactic criterion for available expressions
- Constraint system to express it
  - Yet to come: Link between CS and path-based criterion
- Naive fixpoint iteration to compute maximum solution of CS
- Partial orders, complete lattices

Monotonic function

Definition

Let $(\mathbb{D}_1, \sqsubseteq_1)$ and $(\mathbb{D}_2, \sqsubseteq_2)$ be partial orders. A function $f : \mathbb{D}_1 \to \mathbb{D}_2$ is called monotonic, iff

$$\forall x, y \in \mathbb{D}_1 .\ x \sqsubseteq_1 y \implies f(x) \sqsubseteq_2 f(y)$$

Examples

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  - Functions involving negation/complement usually not monotonic.

Least fixed point

Definition

Let $f : \mathbb{D} \to \mathbb{D}$ be a function.
A value $d \in \mathbb{D}$ with $f(d) = d$ is called fixed point of f.

If $\mathbb{D}$ is a partial ordering, a fixed point $d_0 \in D$ with

$$\forall d.\ f(d) = d \implies d_0 \sqsubseteq d$$

is called least fixed point. If such a d_0 exists, it is uniquely determined, and we define

$$\mathrm{lfp}(f) := d_0$$

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- $f :: 2^{\{a,b,c\}} \to 2^{\{a,b,c\}}$ with $f(X) = X \cup \{a, b\}$. $\mathrm{lfp}(f) = \{a, b\}$.

Function composition

Theorem

If $f_1 : \mathbb{D}_1 \to \mathbb{D}_2$ *and* $f_2 : \mathbb{D}_2 \to \mathbb{D}_3$ *are monotonic, then also* $f_2 \circ f_1$ *is monotonic.*

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Proof: $a \sqsubseteq b \implies f_1(a) \sqsubseteq f_1(b) \implies f_2(f_1(a)) \sqsubseteq f_2(f_1(b))$.

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If $(\mathbb{D}, \sqsubseteq)$ is a partial ordering/complete lattice, then also $([A \to \mathbb{D}], \sqsubseteq)$.
In particular, we have:

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Proof: On whiteboard.

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- Tuples $\vec{x} \in \mathbb{D}^n$ can be seen as functions $\vec{x} : \{1, \ldots, n\} \to \mathbb{D}$

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where

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$(\mathbb{D}, \sqsubseteq)$	complete lattice	e.g., $(2^{\mathrm{Expr}}, \supseteq)$
$f_i : \mathbb{D}^n \to \mathbb{D}$	RHS	e.g., $(A[u] \cup \{e\}) \setminus \mathrm{Expr}_R$

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  - Replace by $x_i \sqsupseteq (\bigsqcup\{rhs_j \mid 1 \le j \le m\})(x_1, \dots, x_n)$

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- Observation: One constraint per x_i is enough.
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- Idea: Encode constraint system as function. Solutions as fixed points.
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- Fixed-Points of F are solutions
- Least solution = least fixed point (next!)

Least fixed points of monotonic functions

- Moreover, F is monotonic if the f_i are.
- Question: Does $\mathrm{lfp}(F)$ exist? Does fp-iteration compute it?

Knaster-Tarski fixed-point Theorem

Knaster-Tarski

Let $(\mathbb{D}, \sqsubseteq)$ *be a complete lattice, and* $f : \mathbb{D} \to \mathbb{D}$ *be a monotonic function. Then,* f *has a least and a greatest fixed point given by*

$$lfp(f) = \bigsqcap \{x \mid f(x) \sqsubseteq x\} \qquad gfp(f) = \bigsqcup \{x \mid x \sqsubseteq f(x)\}$$

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- Show (2): $\bigsqcap P \sqsubseteq f(\bigsqcap P)$
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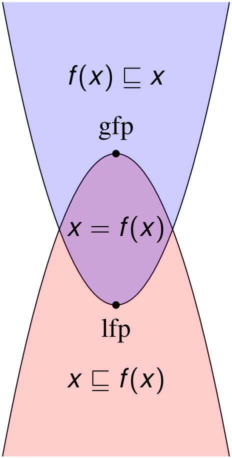
  - Thus $\bigsqcap P \sqsubseteq d$ (lower bound)
- Greatest fixed point: Dually.

□

Used Facts

lower bound	$x \in X \implies \bigsqcap X \sqsubseteq x$
greatest lower bound	$(\forall x \in X.\, d \sqsubseteq X) \implies d \sqsubseteq \bigsqcap X$
mono	f monotonic: $x \sqsubseteq y \implies f(x) \sqsubseteq f(y)$
reflexive	$x \sqsubseteq x$

Knaster-Tarski Fixed-Point Theorem (Intuition)



pre-fixpoints

post-fixpoints

Least solution = lfp

Recall: Constraints where $\vec{x} \sqsupseteq F(\vec{x})$

Knaster-Tarski: $\mathrm{lfp}(F) = \bigsqcap\{\vec{x} \mid \vec{x} \sqsupseteq F(\vec{x})\}$

- I.e.: Least fixed point is lower bound of solutions

Kleene fixed-point theorem

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Let $(\mathbb{D}, \sqsubseteq)$ *be a complete lattice, and* $f : \mathbb{D} \to \mathbb{D}$ *be a monotonic function. Then:*

$$\bigsqcup \{f^i(\bot) \mid i \in \mathbb{N}\} \sqsubseteq \mathit{lfp}(f)$$

If f *is distributive, we even have:*

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Definition

Distributivity A function $f : \mathbb{D}_1 \to \mathbb{D}_2$ over complete lattices $(\mathbb{D}_1, \sqsubseteq_1)$ and $(\mathbb{D}_2, \sqsubseteq_2)$ is called distributive, iff

$$X \neq \emptyset \implies f(\bigsqcup_1 X) = \bigsqcup_2 \{f(x) \mid x \in X\}$$

Note: Distributivity implies monotonicity.

Kleene fixed-point theorem: Proof

By Knaster-Tarski theorem, $\mathrm{lfp}(f)$ exists.

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Thus, $\bigsqcup\{f^i(\bot) \mid i \in \mathbb{N}\} \sqsubseteq \mathrm{lfp}(f)$ (least upper bound) $\square$

Kleene fixed-point theorem: Proof (ctd)

Assume f is distributive.

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With distributive implies mono, antisymmetry and first part, we get:

$$\mathrm{lfp}(f) = \bigsqcup\{f^i(\bot) \mid i \in \mathbb{N}\} \quad \square$$

Used Facts

bot least $\forall x.\ \bot \sqsubseteq x$

fixed point d is fixed point iff $f(d) = d$

least fixed point $f(d) = d \implies \mathrm{lfp}(f) \sqsubseteq d$

least upper bound $(\forall x \in X.\ x \sqsubseteq d) \implies \bigsqcup X \sqsubseteq d$

Summary

- Does $\mathrm{lfp}(F)$ exist?
  - Yes (Knaster-Tarski)

Summary

- Does $\mathrm{lfp}(F)$ exist?
  - Yes (Knaster-Tarski)
- Does fp-iteration compute it?
  - Fp-iteration computes the $F^i(\perp)$ for increasing i
    - By Kleene FP-Theorem, these are below $\mathrm{lfp}(F)$
  - It terminates only if a fixed-point has been reached
    - This fixed point is also below $\mathrm{lfp}(F)$ (and thus = $\mathrm{lfp}(F)$)

Note

- For any monotonic function f, we have

  $$f^{i}(\perp) \sqsubseteq f^{i+1}(\perp)$$

  - Straightforward induction on i

Table of Contents

Naive FP-iteration, again

Input Constraint system $x_i \sqsupseteq f_i(\vec{x})$

1. $\vec{x} := (\bot, \dots, \bot)$
2. $\vec{x} := F(\vec{x})$ (Recall $F(\vec{x}) = (f_1(\vec{x}), \dots, f_n(\vec{x}))$)
3. If $\neg(F(\vec{x}) \sqsubseteq \vec{x})$, goto 2
4. Return "$\vec{x}$ is least solution"

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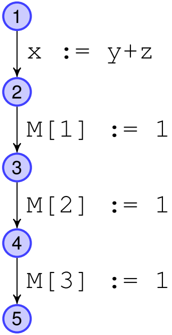
Note Originally, we had $\vec{x} := \vec{x} \sqcup F(\vec{x})$ in Step 2 and $F(\vec{x}) \neq \vec{x}$ in Step 3

- Also correct, as $F^i(\bot) \leq F^{i+1}(\bot)$, i.e., $\vec{x} \sqsubseteq F(\vec{x})$
- Saves $\sqcup$ operation.
- $\sqsubseteq$ may be more efficient than $=$.

Caveat

Naive fp-iteration may be rather inefficient



Let $S := (\text{Expr} \cup \{y + z\}) - \text{Expr}_x$

	0
$A[1]$	Expr
$A[2]$	Expr
$A[3]$	Expr
$A[4]$	Expr
$A[5]$	Expr

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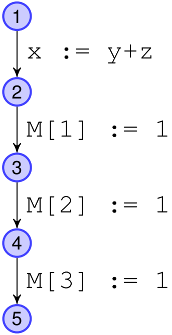

Let $S := (\mathrm{Expr} \cup \{y + z\}) - \mathrm{Expr}_x$

	0	1
$A[1]$	Expr	$\emptyset$
$A[2]$	Expr	S
$A[3]$	Expr	Expr
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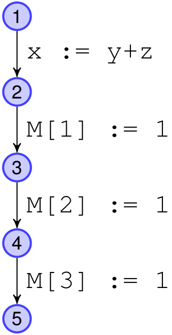

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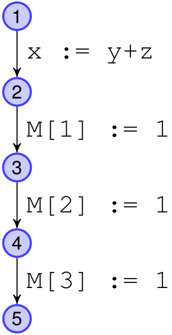

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	0	1	2	3
$A[1]$	Expr	$\emptyset$	$\emptyset$	$\emptyset$
$A[2]$	Expr	S	$\{y + z\}$	$\{y + z\}$
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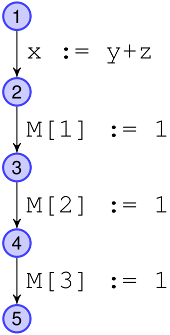
Naive fp-iteration may be rather inefficient



Let $S := (\mathrm{Expr} \cup \{y + z\}) - \mathrm{Expr}_x$

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$A[2]$	Expr	S	$\{y + z\}$	$\{y + z\}$	$\{y + z\}$
$A[3]$	Expr	Expr	S	$\{y + z\}$	$\{y + z\}$
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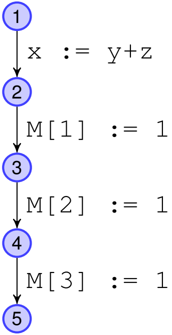

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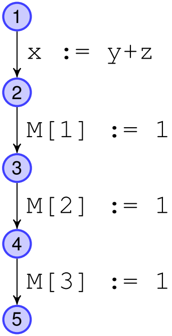
Round-Robin iteration

Idea: Instead of values from last iteration, use current values while computing RHSs.



	0
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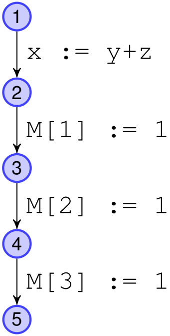

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RR-Iteration: Pseudocode

```
x⃗ := (⊥, ..., ⊥)
do {
  finished := true
  for (i=1;i<=n;++i) {
    new := f_i(x⃗) // Evaluate RHS
    if (x_i ≠ new) { // If something changed
      finished = false    // No fp reached yet
      x_i := x_i ⊔ new // Update variable
    }
  }
} while (!finished)
return x⃗
```

RR-Iteration: Correctness

Prove invariant: $\vec{x} \sqsubseteq \mathrm{lfp}(F)$

- Initially, $(\bot, \ldots, \bot) \sqsubseteq \mathrm{lfp}(F)$ holds (bot-least)

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- Thus: $\vec{x} = \mathrm{lfp}(F)$

Used Facts

trans $x \sqsubseteq y \sqsubseteq z \implies x \sqsubseteq z$

RR-Iteration: Improved Algorithm

We can save some operations

- Use $\sqsubseteq$ instead of $=$ in test
- No $\sqcup$ on update

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With this invariant, we have

- $x_i = f_i(\vec{x})$ iff $x_i \sqsupseteq f_i(\vec{x})$ (antisym)
- $x_i \sqcup f_i(\vec{x}) = f_i(\vec{x})$ (sup-absorb)
  - sup-absorb: $x \sqsubseteq y \implies x \sqcup y = y$

RR-Iteration: Termination

Definition (Chain)

A set $C \subseteq \mathbb{D}$ is called chain, iff all elements are mutually comparable:

$$\forall c_1, c_2 \in C.\ c_1 \sqsubseteq c_2 \vee c_2 \sqsubseteq c_1$$

A partial order has finite height, iff every chain is finite. Then, the height $h \in \mathbb{N}$ is the maximum cardinality of any chain.

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For a domain with finite chain height h, RR-iteration terminates within $O(n^2 h)$ RHS-evaluations.

- In each iteration of the outer loop, at least one variable increases, or the algorithm terminates. A variable may only increase $h - 1$ times.

Last Lecture

- Monotonic functions
  - Constraint system modeled as function
  - Least solution is least fixed point
- Knaster-Tarski fp-thm:
  - lfp of monotonic function exists
- Kleene fp theorem:
  - Iterative characterization of lfp for distributive functions
  - Justifies naive fp-iteration
- Round-Robin iteration
  - Improves on naive iteration by using values of current round
  - Still depends on variable ordering

Problem:

The efficiency of RR depends on variable ordering

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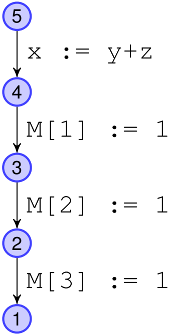

Let $S := (\mathrm{Expr} \cup \{y+z\}) - \mathrm{Expr}_x$

	0	1
$A[1]$	Expr	Expr
$A[2]$	Expr	Expr
$A[3]$	Expr	Expr
$A[4]$	Expr	S
$A[5]$	Expr	$\emptyset$

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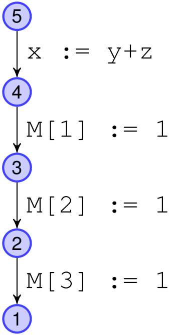

Let $S := (\mathrm{Expr} \cup \{y+z\}) - \mathrm{Expr}_x$

	0	1	2
$A[1]$	Expr	Expr	Expr
$A[2]$	Expr	Expr	Expr
$A[3]$	Expr	Expr	S
$A[4]$	Expr	S	$\{y+z\}$
$A[5]$	Expr	$\emptyset$	$\emptyset$

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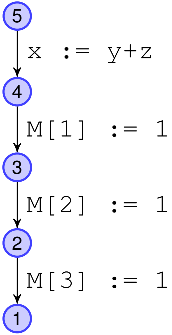

Let $S := (\mathrm{Expr} \cup \{y+z\}) - \mathrm{Expr}_x$

	0	1	2	3
$A[1]$	Expr	Expr	Expr	Expr
$A[2]$	Expr	Expr	Expr	S
$A[3]$	Expr	Expr	S	$\{y+z\}$
$A[4]$	Expr	S	$\{y+z\}$	$\{y+z\}$
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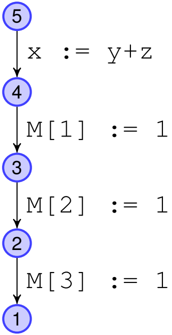

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	0	1	2	3	4
$A[1]$	Expr	Expr	Expr	Expr	S
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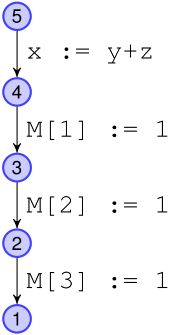

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Problem:

The efficiency of RR depends on variable ordering

Rule of thumb

- u before v, if $u \to^{*} v$
- Entry condition before loop body

Worklist algorithm

Problems of RR (remaining)

- Complete round required to detect termination
- If only one variable changes, everything is re-computed
- Depends on variable ordering.

Worklist algorithm

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Complete round required to detect termination
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Depends on variable ordering.

Idea of worklist algorithm

- Store constraints whose RHS may have changed in a list

Worklist Algorithm: Pseudocode

```
W = {1...n}
x⃗ = (⊥,...,⊥)

while (W != ε) {
  get an i ∈ W, W = W - {i}

  t = f_i(x⃗)
  if (¬(t ⊑ x_i)) {
    x_i = t
    W = W ∪ {j | f_j depends on variable i}
  }
}
```

Worklist Algorithm: Example

- On whiteboard

Worklist Algorithm: Correctness

Invariants

1. $\vec{x} \sqsubseteq F(\vec{x})$ and $\vec{x} \sqsubseteq \mathrm{lfp} F$
   - Same argument as for RR-iteration
2. $\neg(x_i \sqsupseteq f_i(\vec{x})) \implies i \in W$
   - Intuitively: Constraints that are not satisfied are on worklist
   - Initially, all i in W
   - On update: Only RHS that depend on updated variable may change. Exactly these are added to W.
     If f_i does not depend on variable i, the constraint i holds for the new $\vec{x}$, so its removal from W is OK.

- If loop terminates: Due to Inv. 2, we have solution. Due to Inv. 1, it is least solution.

Worklist Algorithm: Termination

Theorem

For a monotonic CS and a domain with finite height h, the worklist algorithm returns the least solution and terminates within $O(hN)$ iterations, where N is the size of the constraint system:

$$N := \sum_{i=1}^{n} 1 + |f_i| \textit{ where } |f_i| := |\{i \mid f_i \textit{ depends on variable } i\}|$$

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Proof (Sketch):

- Number of iterations = Number of elements added to W.
- Initially: n elements
- Constraint i added if variable its RHS depends on is changed
  - Variable may not change more than h times. Constraint depends on $|f_i|$ variables.

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- Number of iterations = Number of elements added to W.
- Initially: n elements
- Constraint i added if variable its RHS depends on is changed
  - Variable may not change more than h times. Constraint depends on $|f_i|$ variables.
- Thus, no more than

  $$n + \sum_{i=1}^{n} h|f_i| = hN$$

  elements added to worklist. $\square$

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- Various more advanced algorithms exists
  - Determine dependencies dynamically (Generic solvers)
  - Only compute solution for subset of the variables (Local solvers)
  - Even: Local generic solvers

Summary:

- Constraint systems (over complete lattice, monotonic RHSs)
  - Encode as monotonic function $F : \mathbb{D}^n \to \mathbb{D}^n$
  - (Least) Solution = (least) fixed point
- Knaster-Tarski theorem: A least solution always exists
- Solve by fixpoint-iteration (naive, RR, WL)
  - Kleene-Theorem justifies naive fixpoint iteration
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- Still Missing:
  - Link between least solution of constraint system, and
    Available at u: $A[u] = \bigcap\{[\![\pi]\!]^\# \emptyset \mid \pi.\ v_0 \xrightarrow{\pi} u\}$

Table of Contents

Monotonic Analysis Framework

Given

Flowgraph

A complete lattice $(\mathbb{D}, \sqsubseteq)$.

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An abstract effect $[\![k]\!]^\# : \mathbb{D} \to \mathbb{D}$ for edges k

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Method Compute least solution MFP of constraint system

$$\mathrm{MFP}[v_0] \sqsupseteq d_0 \qquad \text{(init)}$$

$$\mathrm{MFP}[v] \sqsupseteq [\![k]\!]^\#(\mathrm{MFP}[u]) \quad \text{for edges } k = (u, a, v) \qquad \text{(edge)}$$

MFP = Minimal fixed point

Kam, Ullmann

Kam, Ullman, 1975

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  - $\sqsubseteq \mathrm{MFP}[v]$ ((edge)-constraint, MFP is solution) $\square$

Facts

empty-path	$u \xrightarrow{\varepsilon} v \iff u = v$
empty-eff	$[\![\varepsilon]\!]^{\#} d = d$
eff-comp	$[\![\pi_1 \pi_2]\!]^{\#} = [\![\pi_2]\!]^{\#} \circ [\![\pi_1]\!]^{\#}$

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- Yet another approximation :(
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  - Recall: Abstract effect was already approximation
- Good news:
  - If the right-hand sides are distributive, we can compute MOP exactly

Theorem of Kildal

Kildal, 1972

In a distributive analysis framework (i.e., a monotonic analysis framework where the $[\![k]\!]^\#$ are distributive), where all nodes are reachable, we have

$$MOP = MFP$$

Proof

We already know $\mathrm{MOP} \sqsubseteq \mathrm{MFP}$. To show that also $\mathrm{MFP} \sqsubseteq \mathrm{MOP}$, it suffices to show that MOP is a solution of the constraint system.

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- $= \mathrm{MOP}[v]$ (def.MOP) $\square$

Facts

path-append $k = (u, a, v) \in E \wedge v_0 \xrightarrow{\pi} u \iff v_0 \xrightarrow{\pi k} v$

- Append edge to path

sup-subset $X \subseteq Y \implies \bigsqcup X \sqsubseteq \bigsqcup Y$

Note

Reachability of all nodes is essential

- No paths to unreachable node u, i.e., $\mathrm{MOP}[u] = \bot$
- But edges from other unreachable nodes possible

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Eliminate unreachable nodes before creating CS

- E.g. by DFS from start node.

Depth first search (pseudocode)

```
void dfs (node u) {
  if u ∉ R {
    R := R ∪ {u}
    for all v with (u,a,v) ∈ E {dfs v}
  }
}

void find_reachable () {
  R = {}
  dfs(v0)
  // R contains reachable nodes now
}
```

Summary

Input CFG, distributive/(monotonic) analysis framework

- Framework defines domain $(\mathbb{D}, \sqsubseteq)$, initial value $d_0 \in \mathbb{D}$ and abstract effects $[\![\cdot]\!]^\# : E \to \mathbb{D} \to \mathbb{D}$
- For each edge k, $[\![k]\!]^\#$ is distributive/(monotonic)

1. Eliminate unreachable nodes
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Note Abstract effects of available expressions are distributive

- As all functions of the form: $x \mapsto (a \cup x) \setminus b$

Last lecture

- Worklist algorithm: Find least solution with $O(hN)$ RHS-evaluations
  - h height of domain, N size of constraint system
- Monotonic analysis framework: $(\mathbb{D}, \subseteq)$, $d_0 \in D$, $[\![\cdot]\!]^\#$ (monotonic)
  - Yields $\mathrm{MOP}[u] = \bigsqcup\{[\![\pi]\!]^\# d_0 \mid \pi.\ v_0 \xrightarrow{\pi} u\}$
- Theorems of Kam/Ullman and Kildal
  - $\mathrm{MOP} \sqsubseteq \mathrm{MFP}$,
  - Distributive framework and all nodes reachable: $\mathrm{MOP} = \mathrm{MFP}$
- Started with dead-assignment elimination

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Table of Contents

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- `x` is called dead at 1.

Live registers (semantically)

Register x is semantically live at program point u, iff there is an execution to an end node, that depends on the value of x at u:

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x \in \mathrm{Live}[u] \iff \exists \pi, v, \rho, \mu, a.\ & u \xrightarrow{\pi} v \wedge v \in V_{\mathrm{end}} \\
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  - Equal on memory and "interesting" registers X
- x is semantically dead at u, iff it is not live.
  - No execution depends on the value of x at u.

Transformation: Dead-Assignment Elimination

- Replace assignments/loads to dead registers by `Nop`
- $(u, x := *, v) \mapsto (u, \mathrm{Nop}, v)$ if x dead at v
- Obviously correct
  - States reachable at end nodes are preserved
- Correct approximation: Less dead variables (= More live variables)

Live registers (syntactic approximation)

Register x is live at u ($x \in L[u]$), iff there is a path $u \xrightarrow{\pi} v$, $v \in V_{\mathrm{end}}$, such that

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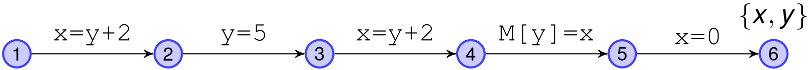
Live at u (MOP): $L[u] = \bigcup \{ [\![\pi]\!]^{\#} X \mid \exists v \in V_{\mathrm{end}}.\ u \xrightarrow{\pi} v \}$

Example

$$1 \xrightarrow{\texttt{x=y+2}} 2 \xrightarrow{\texttt{y=5}} 3 \xrightarrow{\texttt{x=y+2}} 4 \xrightarrow{\texttt{M[y]=x}} 5 \xrightarrow{\texttt{x=0}} 6$$

Example



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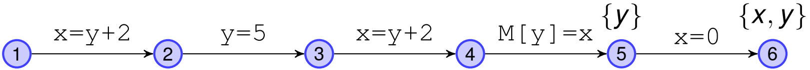

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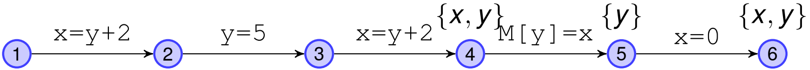

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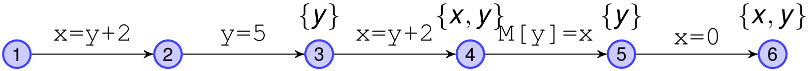

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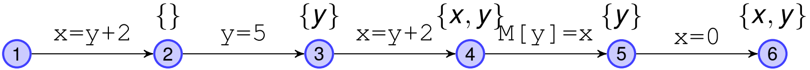

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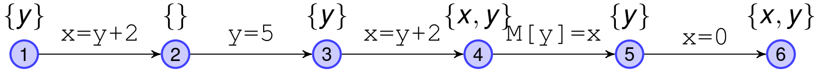

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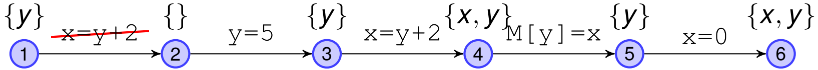

Liveness: Correct approximation

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(Syntactic) liveness is a correct approximation of semantic liveness
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- Proof: On whiteboard.

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Use constraint system

$$L[u] \supseteq X \qquad \text{for } u \in V_{\text{end}}$$

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- Reg: The finitely many registers occurring in program.
  $\Longrightarrow$ Finite height
- Moreover, the $[\![k]\!]^{\#}$ are distributive

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Can compute least solution (MFP)

- Worklist algo, RR-iteration, naive fp-iteration

Backwards Analysis Framework

Given CFG, Domain: $(\mathbb{D}, \sqsubseteq)$, init. value: $d_0 \in \mathbb{D}$, abstract effects: $[\![\cdot]\!]^\# : \mathbb{D} \to \mathbb{D}$, monotonic

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- Proofs:
  - Analogously to forward case :) $\square$

Example: Dead Assignment elimination

```
while (x>0) {
  y = y + 1
  x = x + y
  x = 1
}
```

On whiteboard.

Last Lecture

- Monotonic forward/backward framework
- Live variables, dead assignment elimination
  - x live at u
  - Semantically: $x \in \mathrm{Live}[u]$: Exists execution that depends on value of x at u
  - Syntactic approximation: $x \in L[u]$: x read before it is overwritten
  - Correctness proof
    - Induction on path, case distinction over edges

Analysis: Classifications

- Forward vs. backward

Forward	Considers executions reaching a program point
Backwards	Considers executions from program point to end

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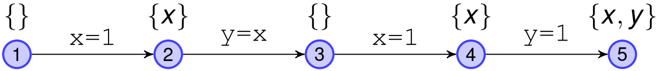
Dead Assignment Elimination: Problems

Eliminating dead assignments may lead to new dead assignments



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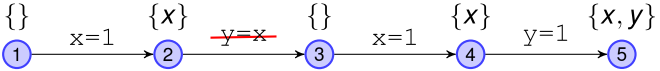

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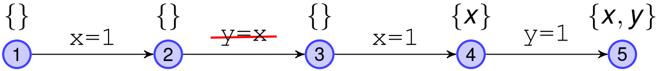

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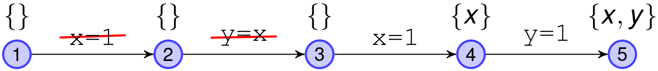

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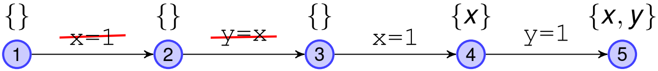
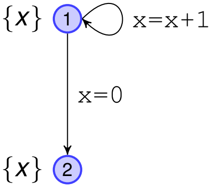


In a loop, a variable may keep itself alive

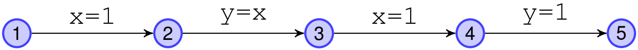


Truly live registers

Idea: Consider assignment edge $(u, x = e, v)$.

- If x is not semantically live at v, the registers in e need not become live at u
- There values influence a register that is dead anyway.

Example



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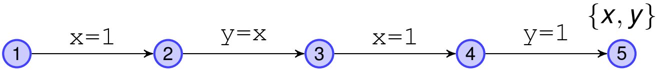

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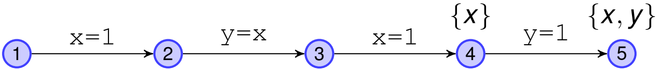

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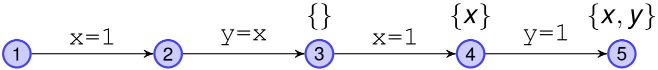

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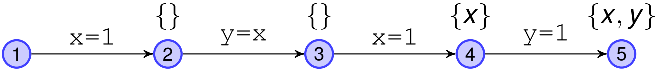

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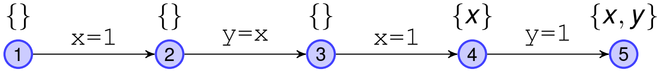

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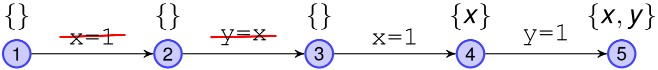
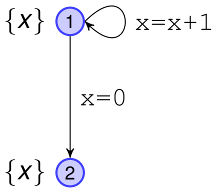


True Liveness vs. repeated liveness

- True liveness detects more dead variables than repeated liveness

Repeated liveness:



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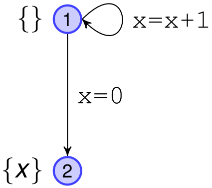

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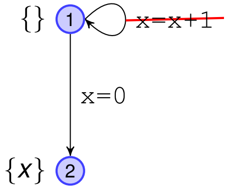

Live registers: Abstract effects

$$
\begin{array}{rll}
[\![\mathrm{Nop}]\!]^\# & L = & L \\
[\![\mathrm{Pos}(e)]\!]^\# & L = & L \cup \mathrm{regs}(e) \\
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We have $\text{MFP} = \text{MOP}$:)

True Liveness: Correct approximation

Theorem

True liveness is a correct approximation of semantic liveness $\mathit{Live}[u] \subseteq TL[u]$

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- Proof: On whiteboard.

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Transformation: Replace variables in expressions according to graph

Example

On Whiteboard

Abstract Effects

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\begin{aligned}
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[\![x = y]\!]^{\#} C &= C \setminus \{x \to *, * \to x\} \cup \{x \to y\} && \text{for } y \in \mathrm{Reg}, y \neq x \\
[\![x = e]\!]^{\#} C &= C \setminus \{x \to *, * \to x\} && \text{for } e \in \mathrm{Expr} \setminus \mathrm{Reg} \text{ or } e = x \\
[\![x = M[e]]\!]^{\#} C &= C \setminus \{x \to *, * \to x\} \\
[\![M[e_1] = e_2]\!]^{\#} C &= C
\end{aligned}
$$

where $\{x \to *, * \to x\}$ is the set of edges from/to x

Obviously, abstract effects preserve acyclicity of C

Moreover, out-degree of nodes is ≤ 1

Abstract effects are distributive

Last Lecture

- Classification of analysis
  - Forward vs. backward, must vs. may, kill/gen, bitvector
- Truly live variables
  - Better approximation of „semantically life”
  - Idea: Don’t care about values of variables that only affect dead variables anyway.
- Copy propagation
  - Replace registers by registers with equal value, to create dead assignments
- Whole procedure: Simple redundancy elimination, then CP and DAE to clean up

Analysis Framework

- Domain: $(\mathbb{D} = 2^{\mathrm{Reg}\times\mathrm{Reg}}, \supseteq)$
  - I.e.: More precise means more edges (Safe approximation: less edges)
  - Join: $\cap$ (Must analysis)
  - Forward analysis, initial value $d_0 = \emptyset$

$\Longrightarrow \quad \mathrm{MOP}[u] = \bigcap\{[\![\pi]\!]^{\#}\emptyset \mid v_0 \xrightarrow{\pi} u\}$

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  - Intuition: $\bot$ means unreachable.

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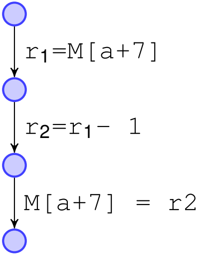
Procedure as a whole

1. Simple redundancy elimination
   - Replaces re-computation by memorization
   - Inserts superfluous moves
2. Copy propagation
   - Removes superfluous moves
   - Creates dead assignments
3. Dead assignment elimination

Example: $a[7] - -$



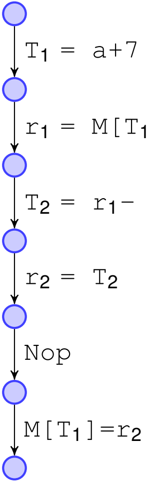
Example: $a[7] - -$

Introduced memorization registers

```
T1 = a+7
r1 = M[T1]
T2 = r1- 1
r2 = T2
T1 = a+7
M[T1]=r2
```

Example: $a[7]--$

Eliminated redundant computations



Example: $a[7]--$

Copy propagation done

$\mathtt{T}_1 = \mathtt{a+7}$

$\mathtt{r}_1 = \mathtt{M[T}_1\mathtt{]}$

$\mathtt{T}_2 = \mathtt{r}_1\mathtt{-\ 1}$

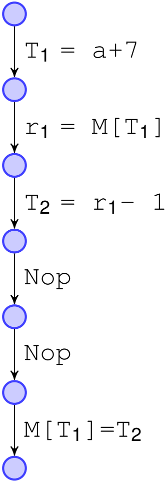
$\mathtt{r}_2 = \mathtt{T}_2$

Nop

$\mathtt{M[T}_1\mathtt{]=T}_2$

Example: $a[7]--$

Eliminated dead assignments



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    - With forward-generalization to paths: $[\![k_1 \dots k_n]\!] = [\![k_n]\!] \circ \dots \circ [\![k_1]\!]$ and $[\![k_1 \dots k_n]\!]^\# = [\![k_n]\!]^\# \circ \dots \circ [\![k_1]\!]^\#$

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- Proof: Straightforward induction on paths. On whiteboard!

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- Now: $c \mathrel{\Delta} d$ — Concrete value c described by abstract value d

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$$c \mathrel{\Delta} d \wedge d \sqsubseteq d' \implies c \mathrel{\Delta} d'$$

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- Assume Δ to be distributive on concrete values:

$$(\forall c \in C.\ c \mathrel{\Delta} d) \iff (\bigsqcup C) \mathrel{\Delta} d$$

  - Note: Implies anti-monotonicity: $c' \sqsubseteq c \wedge c \mathrel{\Delta} d \implies c' \mathrel{\Delta} d$
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$$c \,\Delta\, d \wedge d \sqsubseteq d' \implies c \,\Delta\, d'$$

  - Intuition: Less precise abstract value still describes concrete value
- Assume Δ to be distributive on concrete values:

$$(\forall c \in C.\, c \,\Delta\, d) \iff (\bigsqcup C) \,\Delta\, d$$

  - Note: Implies anti-monotonicity: $c' \sqsubseteq c \wedge c \,\Delta\, d \implies c' \,\Delta\, d$
  - Intuition: More precise concrete values still described by abstract value
- We get for all sets of paths P:

$$(\forall \pi \in P.\, [\![\pi]\!] c_0 \,\Delta\, [\![\pi]\!]^{\#} d_0) \implies (\bigsqcup_{\pi \in P} [\![\pi]\!] c_0) \,\Delta\, (\bigsqcup_{\pi \in P} [\![\pi]\!]^{\#} d_0)$$

  - Intuition: Concrete values due to paths P described by abstract values

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- We get: $[\![u]\!] \;\Delta\; \mathrm{MOP}[u]$
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- Get: $[\![u]\!] \,\Delta\, \mathrm{MOP}[u]$, thus $[\![u]\!] \,\Delta\, \mathrm{MFP}[u]$
  - Which justifies correctness of transformation wrt. MFP

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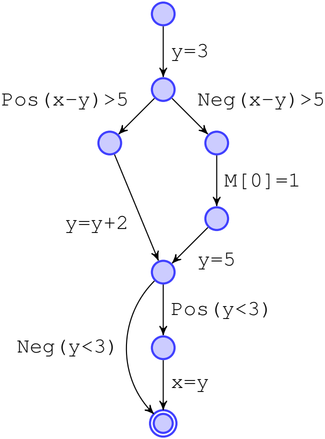

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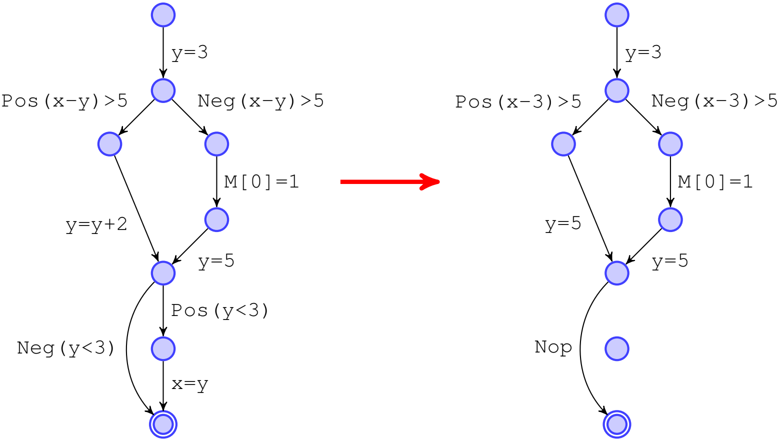

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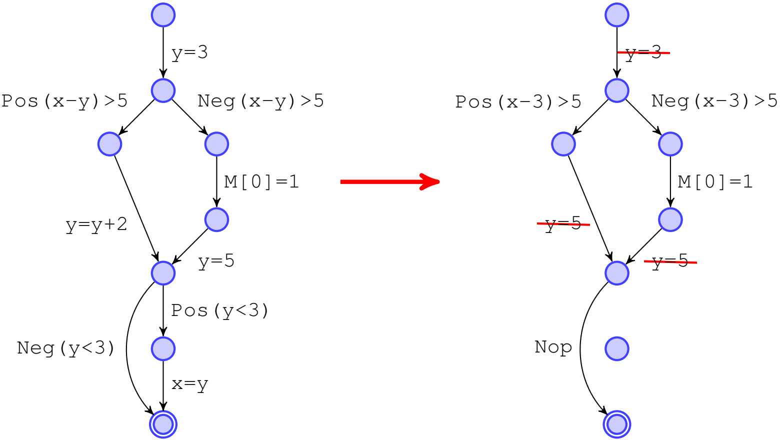

- Dead-code elimination afterwards to clean up (assume *y* not interesting)

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$$
\begin{tikzcd}
 & & & \top & & & \\
\cdots & -2 \arrow[urr, no head] & -1 \arrow[ur, no head] & 0 \arrow[u, no head] & 1 \arrow[ul, no head] & 2 \arrow[ull, no head] & \cdots
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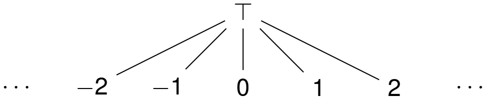

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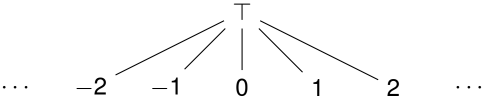

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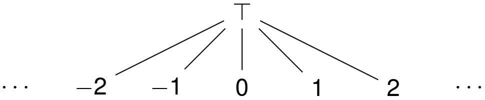

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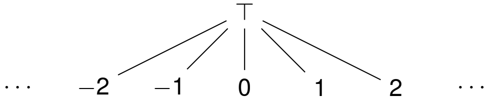

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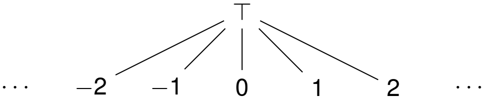

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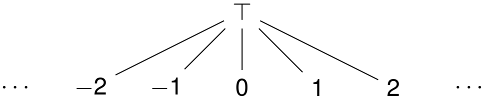

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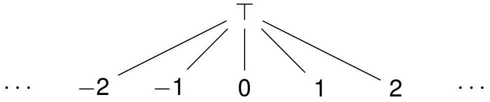

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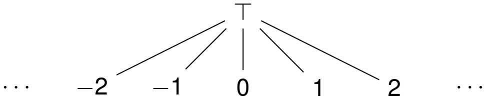

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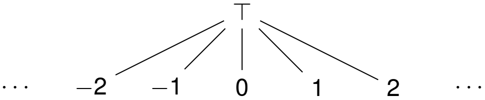

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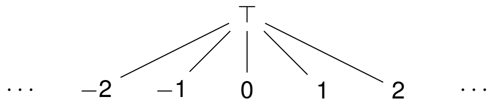
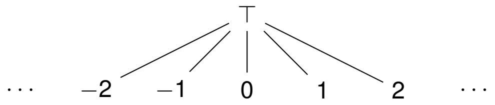


    - Intuition: $\top$ — don't know value of register
  - $\mathbb{D} = (\mathrm{Reg} \to \mathbb{Z}^\top) \cup \{\bot\}$
  - Add a bottom-element
    - Intuition: $\bot$ — program point not reachable
  - Ordering: Pointwise ordering on functions, $\bot$ being the least element.
  - $(\mathbb{D}, \sqsubseteq)$ is complete lattice
- Examples
  - $D[u] = \bot$: u not reachable
  - $D[u] = \{x \mapsto \top, y \mapsto 5\}$:

Approach

- Idea: Store, for each register, whether it is definitely constant at u
  - Assign each register a value from $\mathbb{Z}^\top$



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- Examples
  - $D[u] = \bot$: u not reachable
  - $D[u] = \{x \mapsto \top, y \mapsto 5\}$: y is always 5 at u, nothing known about x

Abstract evaluation of expressions

- For concrete operator $\Box : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$, we define abstract operator $\Box^\# : \mathbb{Z}^\top \times \mathbb{Z}^\top \to \mathbb{Z}^\top$:

$$\top \,\Box^\#\, x := \top$$
$$x \,\Box^\#\, \top := \top$$
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- Evaluate expression wrt. abstract values and operators: $[\![e]\!]^\# : (\mathrm{Reg} \to \mathbb{Z}^\top) \to \mathbb{Z}^\top$

$[\![c]\!]^\# D := c$	for constant c
$[\![r]\!]^\# D := D(r)$	for register r
$[\![e_1 \,\square\, e_2]\!]^\# D := [\![e_1]\!]^\# D \,\square^\#\, [\![e_2]\!]^\# D$	for operator $\square$

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Analogously for unary, ternary, etc. operators

Example

- Example: $D = \{x \mapsto \top, y \mapsto 5\}$

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$$\begin{aligned}[\![y - 3]\!]^\# D &= [\![y]\!]^\# D -^\# [\![3]\!]^\# D \\ &= 5 -^\# 3 \\ &= 2\end{aligned}$$

Example

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$$\begin{aligned} [\![x + y]\!]^{\#} D &= [\![x]\!]^{\#} D +^{\#} [\![y]\!]^{\#} D \\ &= \top +^{\#} 5 \\ &= \top \end{aligned}$$

Abstract effects (forward)

$$\llbracket k \rrbracket^\# \bot := \bot \qquad \text{for any edge } k$$

$$\llbracket \mathrm{Nop} \rrbracket^\# D := D$$

$$\llbracket \mathrm{Pos}(e) \rrbracket^\# := \begin{cases} \bot & \text{if } \llbracket e \rrbracket^\# D = 0 \\ D & \text{otherwise} \end{cases}$$

$$\llbracket \mathrm{Neg}(e) \rrbracket^\# := \begin{cases} \bot & \text{if } \llbracket e \rrbracket^\# D = v, v \in \mathbb{Z} \setminus \{0\} \\ D & \text{otherwise} \end{cases}$$

$$\llbracket r = e \rrbracket^\# D := D(r \mapsto \llbracket e \rrbracket^\# D)$$

$$\llbracket r = M[e] \rrbracket^\# D := D(r \mapsto \top)$$

$$\llbracket M[e_1] = e_2 \rrbracket^\# D := D$$

For $D \neq \bot$.

Abstract effects (forward)

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[\![r = M[e]]\!]^\# D &:= D(r \mapsto \top) \\
[\![M[e_1] = e_2]\!]^\# D &:= D
\end{aligned}
$$

For $D \neq \bot$.

Initial value at start: $d_0 := \lambda x.\ \top$.

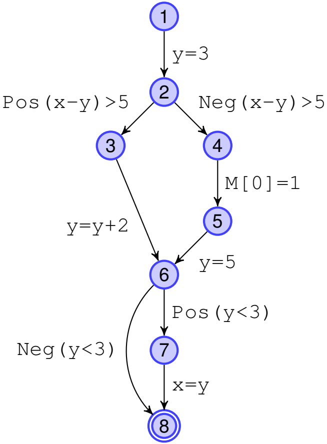
(Reachable, all variables have unknown value)

Last lecture

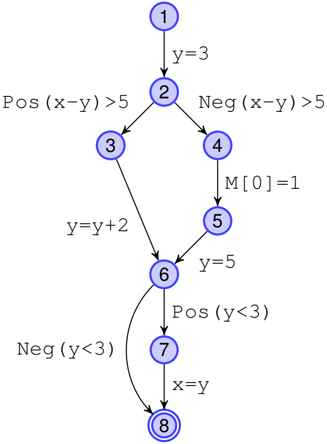
- Simulation based framework for program analysis
- Abstract setting:
  - Actions preserve relation Δ between concrete and abstract state.

    $\Longrightarrow$ States after executing path are related
  - Approximation: Complete lattice structure
    - Δ monotonic
    - Distributive $\Longrightarrow$ generalization to sets of path
- For program analysis:
  - Concrete state: Sets of program states
    - All states reachable via path.
- Constant propagation

Example

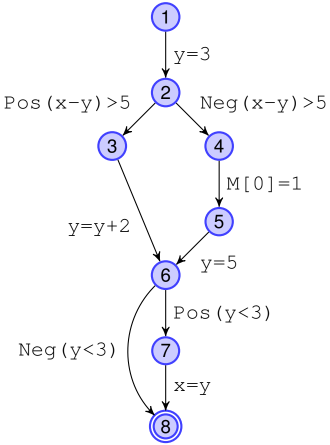


Example



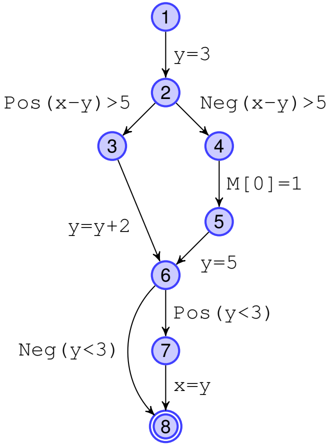
$$\mathrm{D}[1] \;=\; x \mapsto \top, y \mapsto \top$$

Example



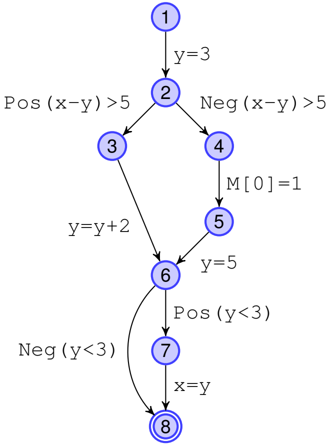
$$
\begin{aligned}
D[1] &= x \mapsto \top, y \mapsto \top \\
D[2] &= x \mapsto \top, y \mapsto 3
\end{aligned}
$$

Example



$$\begin{aligned} D[1] &= x \mapsto \top, y \mapsto \top \\ D[2] &= x \mapsto \top, y \mapsto 3 \\ D[3] &= x \mapsto \top, y \mapsto 3 \end{aligned}$$

Example



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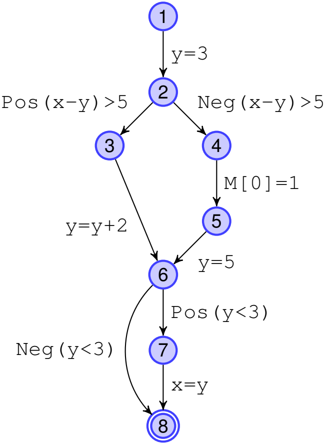
$$

Example



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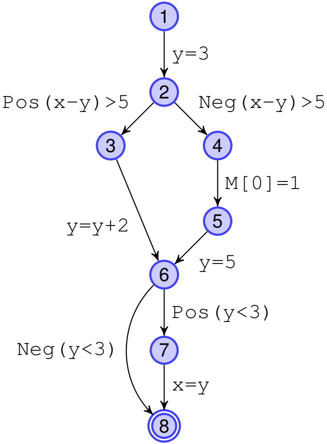

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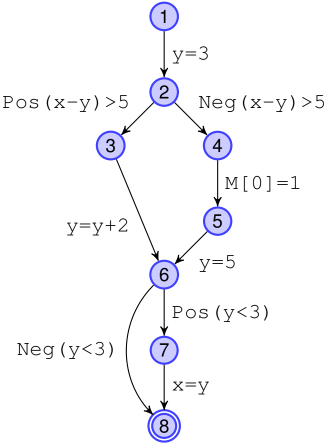

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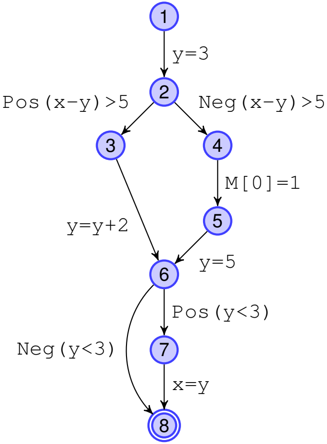
\end{aligned}
$$

Example



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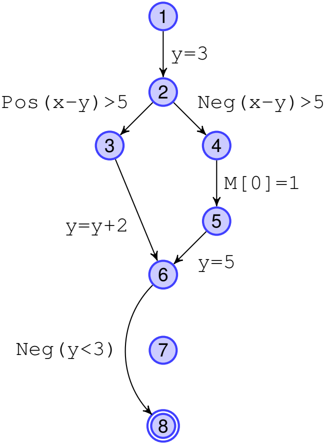

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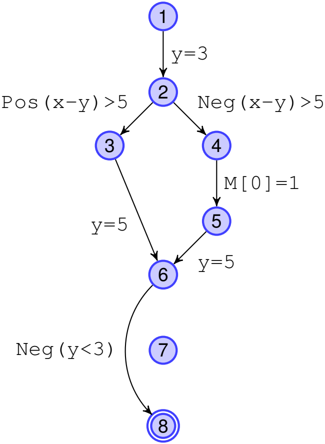
Transformations:

Remove (u, a, v) if $D[u] = \bot$ or $D[v] = \bot$

Example



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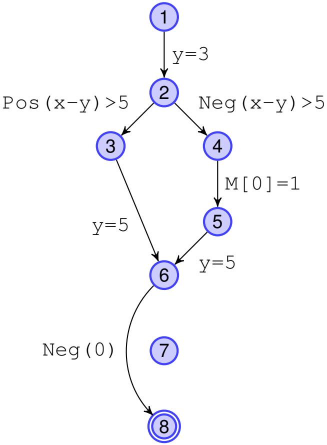
Transformations:

Remove (u, a, v) if $D[u] = \perp$ or $D[v] = \perp$

$(u, r = e, v) \mapsto (u, r = c, v)$ if $[\![e]\!]^{\#}(D[u]) = c \in \mathbb{Z}$

Example



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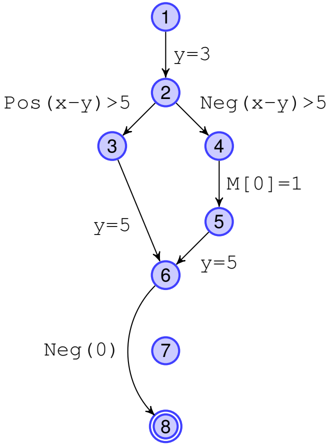
Analogously for test, load, store

Example



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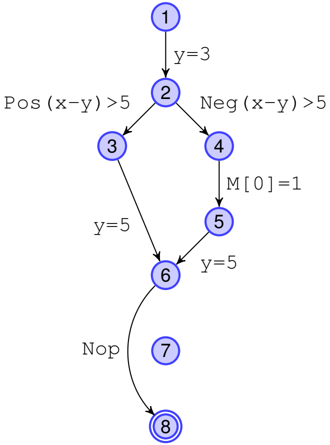
Analogously for test, load, store

$(u, \mathrm{Pos}(c), v) \mapsto \mathrm{Nop}$ if $c \in \mathbb{Z} \setminus \{0\}$

Example



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- Establish description relation
  - Between values, valuations, states

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- Valuations: For $\rho : \mathrm{Reg} \to \mathbb{Z}, \rho^{\#} : \mathrm{Reg} \to \mathbb{Z}^{\top}$: $\rho \ \Delta \ \rho^{\#}$ iff $\forall x.\ \rho(x) \ \Delta \ \rho^{\#}(x)$
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  - Value of each variable must be described.
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- States: $(\rho, \mu) \,\Delta\, \rho^{\#}$ if $\rho \,\Delta\, \rho^{\#}$ and $\forall s.\ \neg(s \,\Delta\, \bot)$
  - Bottom describes no states (i.e., empty set of states)
  - Note: Monotonic. (Only new case: $s \,\Delta\, \bot \wedge \bot \sqsubseteq d \implies s \,\Delta\, d$)

Correctness (Abstract values)

- Show: For every constant c and operator $\Box$, we have

$$c \,\Delta\, c^{\#}$$

$$v_1 \,\Delta\, d_1 \wedge v_2 \,\Delta\, d_2 \implies (v_1 \Box v_2) \,\Delta\, (d_1 \Box^{\#} d_2)$$

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$$\rho \,\Delta\, \rho^{\#} \implies [\![e]\!]\rho \,\Delta\, [\![e]\!]^{\#} \rho^{\#}$$

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- Moreover, show $\forall \rho_0, \mu_0.\ (\rho_0, \mu_0) \,\Delta\, d_0$
  - Here: $\forall \rho_0, \mu_0.\ (\rho_0, \mu_0) \,\Delta\, \lambda x.\ \top$

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$\Longleftarrow\ \rho_0 \;\Delta\; \lambda x.\ \top$

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$\Longleftarrow$ $\rho_0 \,\Delta\, \lambda x.\ \top$

$\Longleftarrow$ $\forall x.\ \rho_0(x) \,\Delta\, \top$. Holds by definition.

Correctness (Of Transformations)

- Assume $(\rho, \mu)\ \Delta\ \rho^{\#}$. Show $[\![a]\!](\rho, \mu) = [\![\mathsf{tr}(a, \rho^{\#})]\!](\rho, \mu)$

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  - Remove edge if $\rho^{\#} = \bot$. Trivial.

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  - Remove edge if $\rho^{\#} = \bot$. Trivial.
  - Replace $r = e$ by $r = [\![e]\!]^{\#} \rho^{\#}$ if $[\![e]\!]^{\#} \rho^{\#} \neq \top$
    - From $\rho \,\Delta\, \rho^{\#} \implies [\![e]\!]\rho \,\Delta\, [\![e]\!]^{\#} \rho^{\#} \implies [\![e]\!]\rho = [\![e]\!]^{\#} \rho^{\#}$

Correctness (Of Transformations)

- Assume $(\rho, \mu) \Delta \rho^\#$. Show $[\![a]\!](\rho, \mu) = [\![\mathrm{tr}(a, \rho^\#)]\!](\rho, \mu)$
  - Remove edge if $\rho^\# = \bot$. Trivial.
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    - Analogously for expressions in load, store, Neg, Pos.
  - Replace tests on constants by `Nop`: Obviously correct.
    - Does not depend on analysis result.

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  - Other cases: Analogously.
- Our general theory gives us: $[\![u]\!] \,\Delta\, \mathrm{MFP}[u]$
  - Thus, transformation wrt. MFP is correct.

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- Thus, MFP only approximation of MOP in general.

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- Proof: By undecidability of Hilbert's 10th problem

Hilbert's 10th problem (1900)

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Theorem (Matiyasevich, 1970)

(Based on work of David, Putnam, Robinson)
It is undecidable whether a Diophantine equation has an integer solution.

Regard the following program

```
x₁=x₂=...xₙ=0
while (*) { x₁ = x₁ + 1 }
...
while (*) { xₙ = xₙ + 1 }
r=0
if (p(x₁,...,xₙ) == 0) then r=1
u: Nop
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- r is only found to be non-constant, if $p(x_1, \ldots, x_n) = 0$
- Thus, $\mathrm{MOP}[u](r) = \top$ if, and only if $p(x_1, \ldots, x_n) = 0$ has a solution $\square$

Extensions

- Also simplify subexpressions:
  - For $\{x \mapsto \top, y \mapsto 3\}$, replace $x + 2 * y$ by $x + 6$.

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    `if` `(x==4) M[0]=5` **`else`** `M[0]=x`

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$$[\![\mathrm{Pos}(x == e)]\!]^{\#} = \begin{cases} D & \text{if } [\![x == e]\!]^{\#} D = 1 \\ \bot & \text{if } [\![x == e]\!]^{\#} D = 0 \\ D_1 & \text{otherwise} \end{cases}$$

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  - Analogously for $\mathrm{Neg}(x \neq e)$

Table of Contents

Interval Analysis

- Constant propagation finds constants
- But sometimes, we can restrict the value of a variable to an interval, e.g., [0..42].

Example

```
int a[42];
for (i=0;i<42;++i) {
  if (0<=i && i<42)
    a[i] = i*2;
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    fail();
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Problems

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Ordering $\subseteq$, i.e. $[l_1, u_1] \sqsubseteq [l_2, u_2]$ iff $l_1 \geq l_2 \wedge u_1 \leq u_2$

- Smaller interval contained in larger one
- Hence:

$$[l_1, u_1] \sqcup [l_2, u_2] = [\min(l_1, l_2), \max(u_1, u_2)]$$
$$\top = [-\infty, +\infty]$$

Problems

- Not a complete lattice. (Will add $\perp$ - element later)

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Problems

- Not a complete lattice. (Will add $\bot$ - element later)
- Infinite ascending chains: $[0, 0] \sqsubset [0, 1] \sqsubset [0, 2] \sqsubset \ldots$

Building the Domain

- Analogously to CP:
  - $\mathbb{D} := (\mathrm{Reg} \to \mathbb{I}) \cup \{\bot\}$
  - Intuition: Map variables to intervals their value must be contained in.
  - $\bot$ — unreachable

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  - Obviously monotonic. (Larger interval admits more values)

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Division $[l_1, u_1]/^\#[l_2, u_2] :=$
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- If $0 \notin [l_2, u_2]$, otherwise $[l_1, u_1]/^\#[l_2, u_2] := \top$

Examples

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Proof obligations

$c \mathrel{\Delta} c^{\#}$

$v_1 \mathrel{\Delta} d_1 \wedge v_2 \mathrel{\Delta} d_2 \implies v_1 \square v_2 \mathrel{\Delta} d_1 \square^{\#} d_2$

Analogously for unary, ternary, etc. operators

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Then, we get $\rho \; \Delta \; \rho^{\#} \implies [\![e]\!]\rho \; \Delta \; [\![e]\!]^{\#}\rho^{\#}$

- As for constant propagation

Effects of edges

For $\rho^\# \neq \bot$

$$
\begin{aligned}
[\![\cdot]\!]^\# \bot &= \bot \\
[\![\mathrm{Nop}]\!]^\# \rho^\# &= \rho^\# \\
[\![x = e]\!]^\# \rho^\# &= \rho^\#(x \mapsto [\![e]\!]^\# \rho^\#) \\
[\![x = M[e]]\!]^\# \rho^\# &= \rho^\#(x \mapsto \top) \\
[\![M[e_1] = e_2]\!]^\# \rho^\# &= \rho^\# \\
[\![\mathrm{Pos}(e)]\!]^\# \rho^\# &= \begin{cases} \bot & \text{if } [\![e]\!]^\# \rho^\# = [0,0] \\ \rho^\# & \text{otherwise} \end{cases} \\
[\![\mathrm{Neg}(e)]\!]^\# \rho^\# &= \begin{cases} \rho^\# & \text{if } [\![e]\!]^\# \rho^\# \sqsupseteq [0,0] \\ \bot & \text{otherwise} \end{cases}
\end{aligned}
$$

Last lecture

- Constant propagation
  - Idea: Abstract description of values, lift to valuations, states
- Monotonic, but not distributive
  - MOP solution undecidable (Reduction to Hilbert's 10th problem)
- Interval analysis
  - Associate variables with intervals of possible values

Better exploitation of conditions

$$
[\![\mathrm{Pos}(e)]\!]^\# \rho^\# = \begin{cases}
\bot & \text{if } [\![e]\!]^\# \rho^\# = [0,0] \\
\rho^\# (x \mapsto \rho^\#(x) \sqcap [\![e_1]\!]^\# \rho^\#) & \text{if } e \ = \ x == e_1 \\
\rho^\# (x \mapsto \rho^\#(x) \sqcap [-\infty, u]) & \text{if } e \ = \ x \leq e_1 \text{ and } [\![e_1]\!]^\# \rho^\# = [\_, u] \\
\rho^\# (x \mapsto \rho^\#(x) \sqcap [l, \infty]) & \text{if } e \ = \ x \geq e_1 \text{ and } [\![e_1]\!]^\# \rho^\# = [l, \_] \\
\ldots & \\
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\end{cases}
$$

$$
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\rho^\# (x \mapsto \rho^\#(x) \sqcap [\![e_1]\!]^\# \rho^\#) & \text{if } e \ = \ x \neq e_1 \\
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$$

- where $[l_1, u_1] \sqcap [l_2, u_2] = [\max(l_1, l_2), \min(u_1, u_2)]$
  - only exists if intervals overlap
  - this is guaranteed by conditions

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- Erase nodes u with $\mathrm{MOP}[u] = \bot$ (unreachable)

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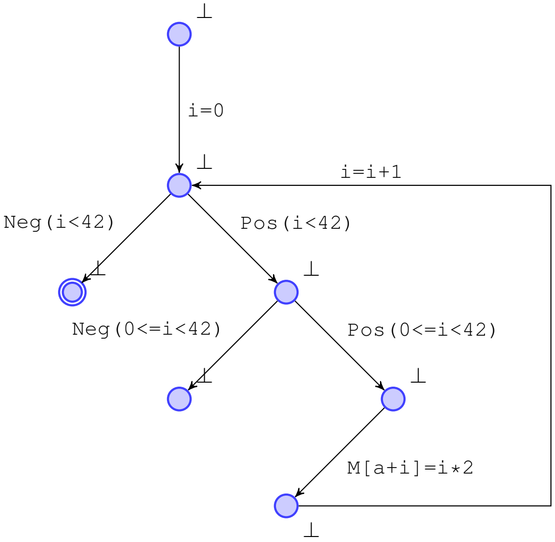
- Transformation: $(u, k, v) \mapsto (u, \mathrm{tr}(k, \mathrm{MFP}[u]), v)$
- Proof obligation:
  - $(\rho, \mu)\ \Delta\ \rho^{\#} \implies [\![k]\!](\rho, \mu) = [\![\mathrm{tr}(k, \rho^{\#})]\!](\rho, \mu)$

Example



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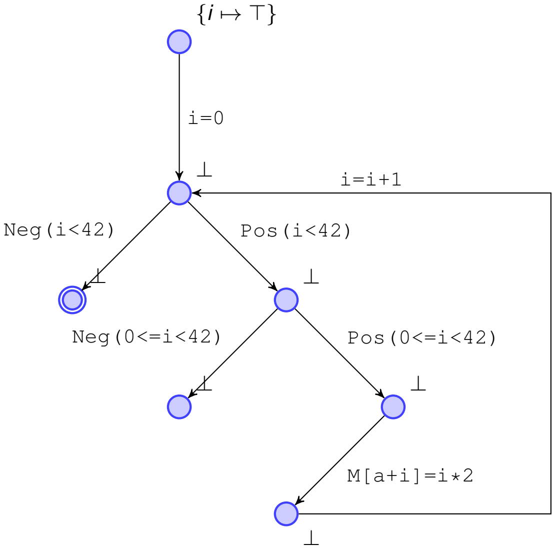
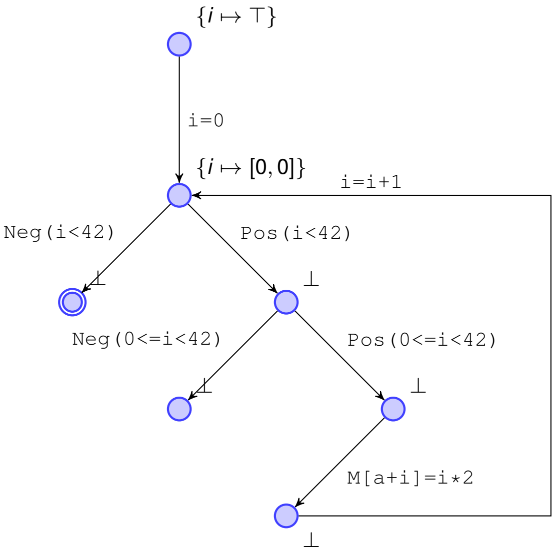


$\{i \mapsto \top\}$
i=0
$\perp$
i=i+1
Neg(i<42)
Pos(i<42)
$\perp$
$\perp$
Neg(0<=i<42)
Pos(0<=i<42)
$\perp$
$\perp$
M[a+i]=i*2
$\perp$

Example



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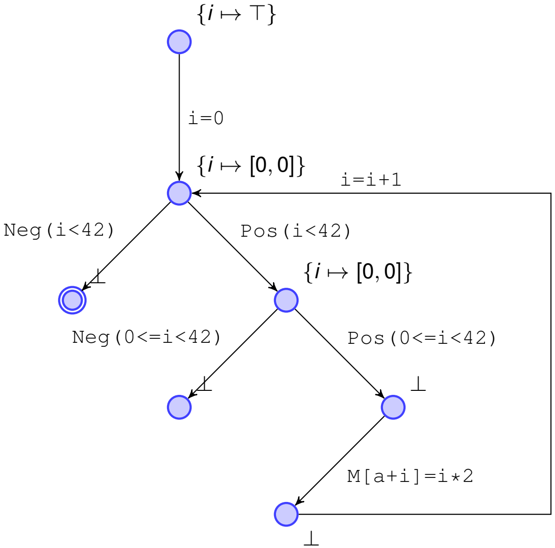

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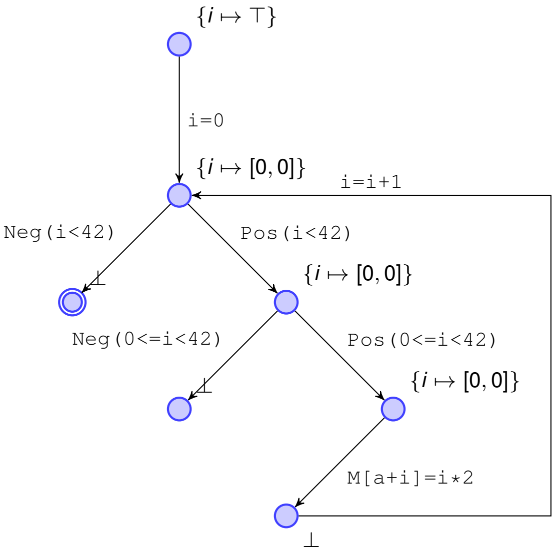
Pos(0<=i<42)
$\perp$
$\perp$
M[a+i]=i*2
$\perp$

Example



$\{i \mapsto \top\}$
i=0
$\{i \mapsto [0,0]\}$
i=i+1
Neg(i<42)
Pos(i<42)
$\bot$
$\{i \mapsto [0,0]\}$
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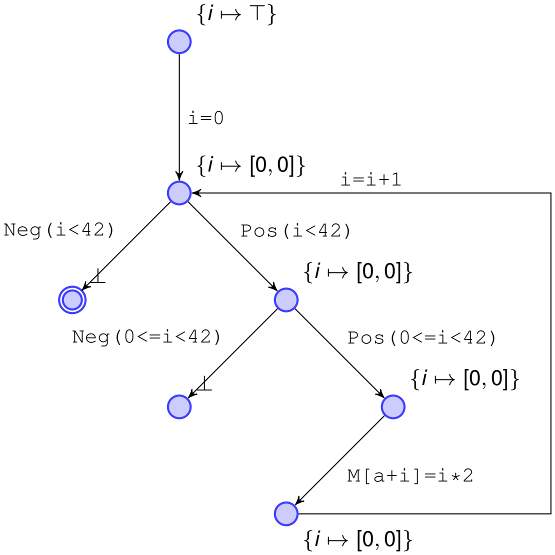

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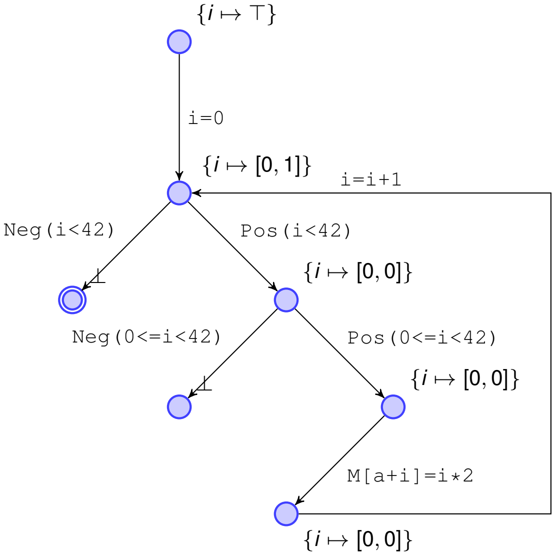
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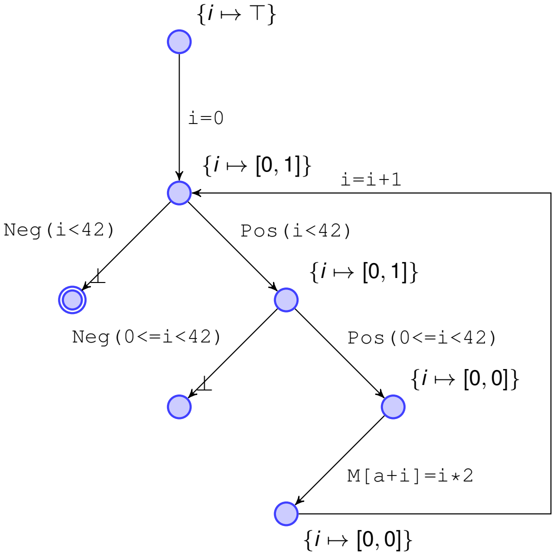

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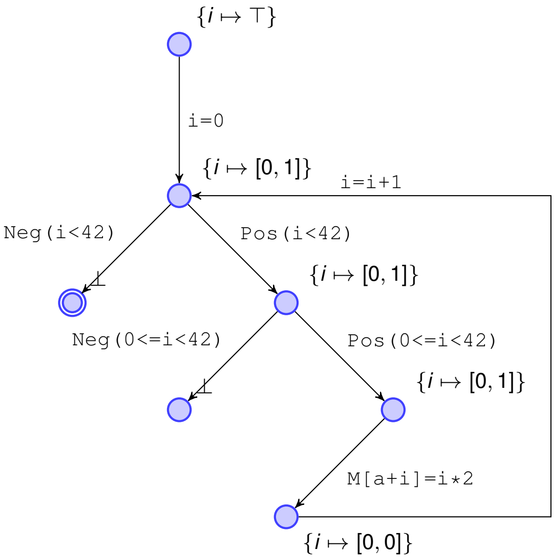

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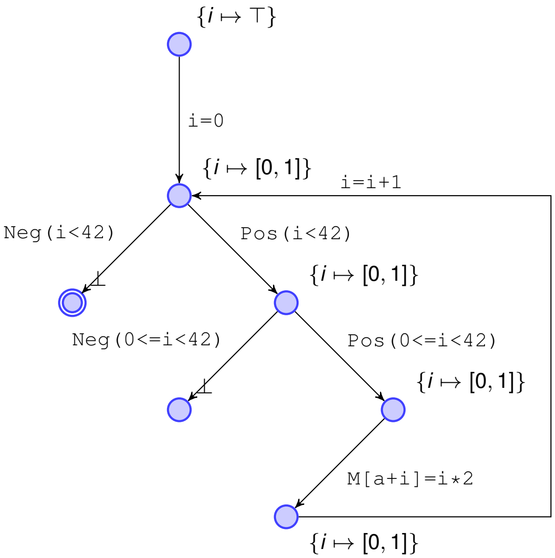

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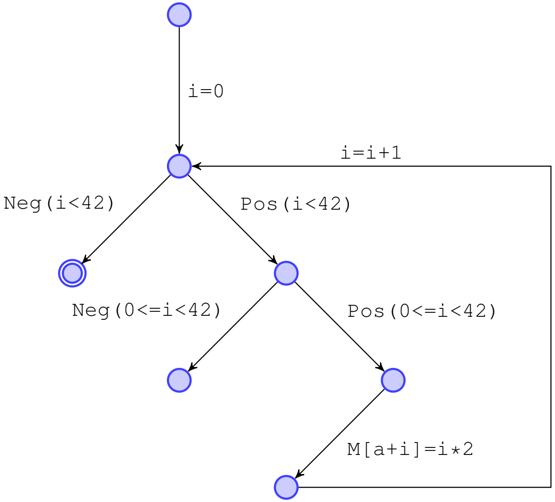
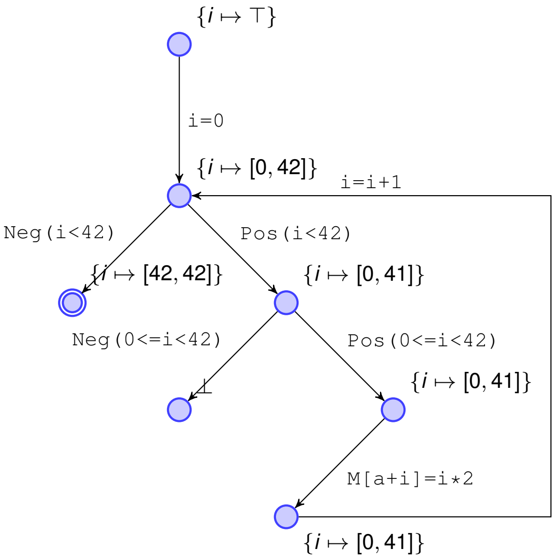


About 40 iterations later ...

Example



About 40 iterations later ...

Problem

- Interval analysis takes many iterations
- May not terminate at all `for (i=0;x>0;x--) i=i+1`

Widening

- Idea: Accelerate the iteration

Widening

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Widening

- Idea: Accelerate the iteration — at the price of imprecision
- Here: Disallow updates of interval bounds in $\mathbb{Z}$.
  - A maximal chain: $[3, 8] \sqsubseteq [-\infty, 8] \sqsubseteq [-\infty, \infty]$

Widening (Formally)

- Given: Constraint system (1) $x_i \sqsupseteq f_i(\vec{x})$
  - f_i not necessarily monotonic
- Regard the system (2) $x_i = x_i \sqcup f_i(\vec{x})$
- Obviously: $\vec{x}$ solution of (1) iff $\vec{x}$ solution of (2)
  - Note: $x \sqsubseteq y \iff x \sqcup y = y$
- (2) induces a function $G : \mathbb{D}^n \to \mathbb{D}^n$

$$G(\vec{x}) = \vec{x} \sqcup (f_1(\vec{x}), \ldots, f_n(\vec{x}))$$

- G is not necessarily monotonic, but increasing:

$$\forall \vec{x}.\ \vec{x} \sqsubseteq G(\vec{x})$$

Widening (Formally)

- G is increasing $\implies \bot \sqsubseteq G(\bot) \sqsubseteq G^2(\bot) \sqsubseteq \dots$
  - i.e., $\langle G^i(\bot)\rangle_{i\in\mathbb{N}}$ is ascending chain
- If it stabilizes, i.e., $\vec{x} = G^k(\bot) = G^{k+1}(\bot)$, then $\vec{x}$ is solution of (1)
- If $\mathbb{D}$ has infinite ascending chains, still no termination guaranteed
- Replace $\sqcup$ by *widening operator* $\sqcup\!\!\!\!\!\sqcup$
  - Get (3) $x_i = x_i \sqcup\!\!\!\!\!\sqcup f_i(\vec{x})$
- Widening: Any operation $\mathbb{D} \times \mathbb{D} \to \mathbb{D}$
  1. with $x \sqcup y \sqsubseteq x \sqcup\!\!\!\!\!\sqcup y$
  2. and for every sequence $a_0, a_1, \dots$, the chain $b_0 = a_0$, $b_{i+1} = b_i \sqcup\!\!\!\!\!\sqcup a_{i+1}$ eventually stabilizes
- Using FP-iteration (naive, RR, worklist) on (3) will
  - compute a solution of (1)
  - terminate

To show

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  - $x_i = x_i \sqcup f_i(\vec{x})$

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  - Valuation increases until it stabilizes (latest at $\vec{x} = (\top, \dots, \top)$)

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- Solutions of (3) are solutions of (1)
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- FP-iteration computes a solution of (3).
  - Valuation increases until it stabilizes (latest at $\vec{x} = (\top, \dots, \top)$)
- FP-iteration terminates
  - FP-iteration step: Replace (some) x_i by $x_i \sqcup\!\!\!\!\sqcup f_i(\vec{x})$
  - This only happens finitely many times (Widening operator, Criterion 2)

For interval analysis

- Widening defined as $[l_1, u_1] \sqcup\!\!\!\!\sqcup [l_2, u_2] := [l, u]$ with

$$l := \begin{cases} l_1 & \text{if } l_1 \leq l_2 \\ -\infty & \text{otherwise} \end{cases}$$

$$u := \begin{cases} u_1 & \text{if } u_1 \geq u_2 \\ +\infty & \text{otherwise} \end{cases}$$

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  2. Lower and upper bound updated at most once.
- Note: $\sqcup\!\!\!\!\!\sqcup$ is not commutative.

Examples

- $[-2, 2] \sqcup [1, 2] =$

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- $[1,2] \sqcup [-2,2] = [-\infty,2]$
- $[1,2] \sqcup [1,3] = [1,+\infty]$
- Widening returns larger values more quickly

Widening (Intermediate Result)

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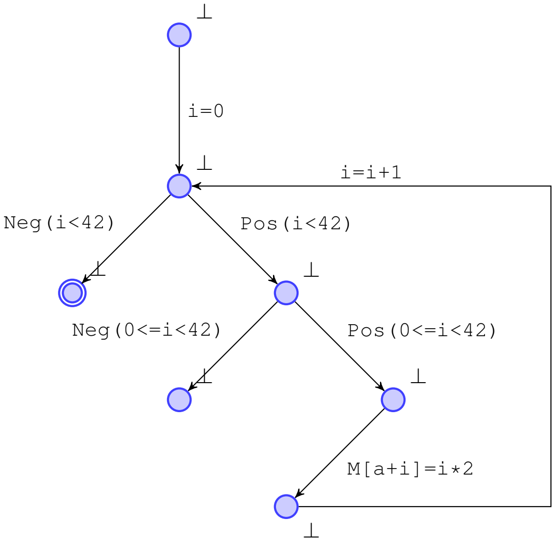
Widening (Intermediate Result)

- Define suitable widening
- Solve constraint system (3)
- Guaranteed to terminate and return over-approximation of MOP
- But: Construction of good widening is black magic
    - Even may choose $\sqcup$ dynamically during iteration, such that
        - Values do not get too complicated
        - Iteration is guaranteed to terminate

Example (Revisited)



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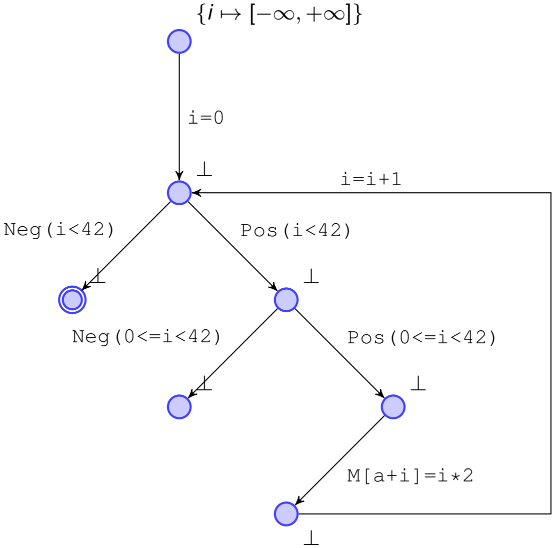

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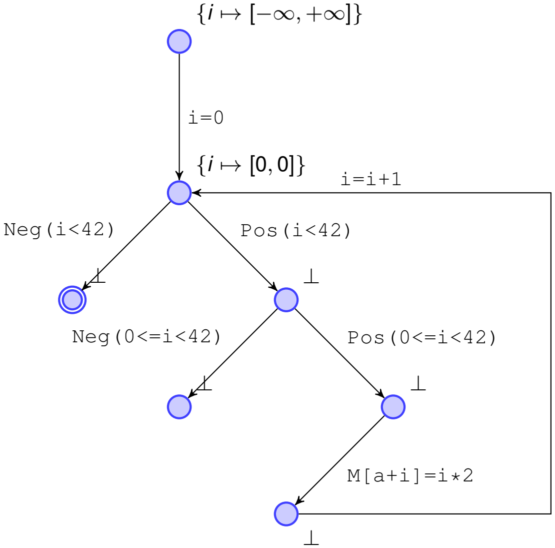

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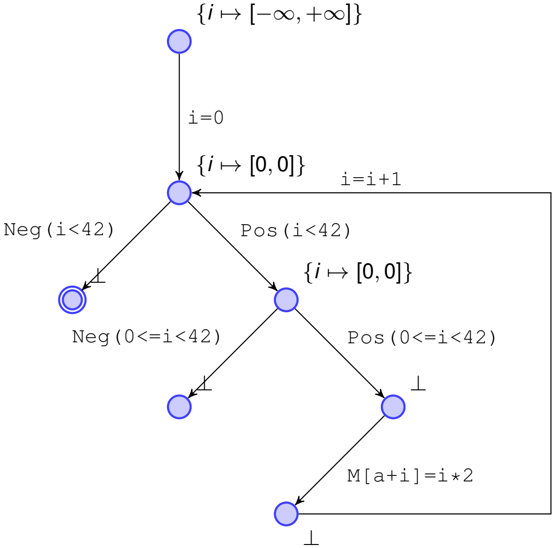

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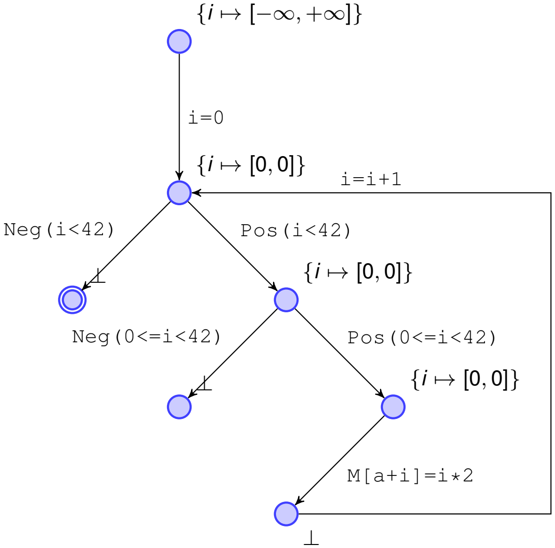

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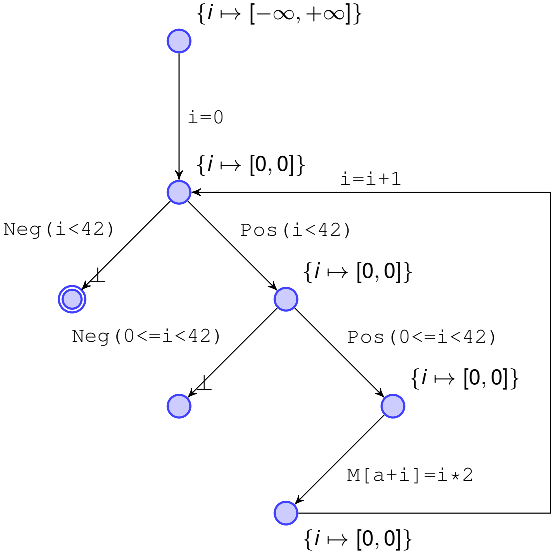

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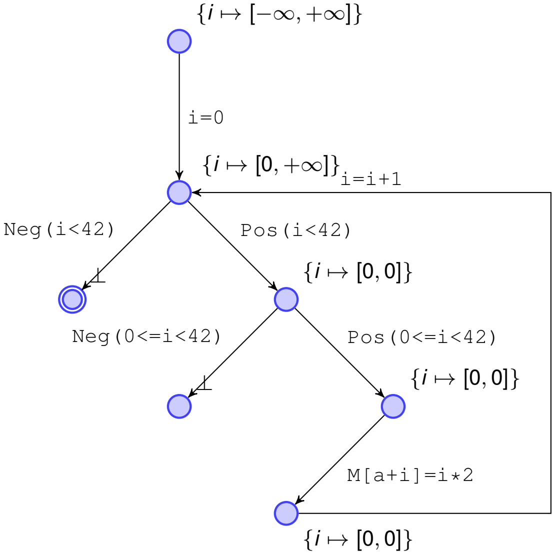

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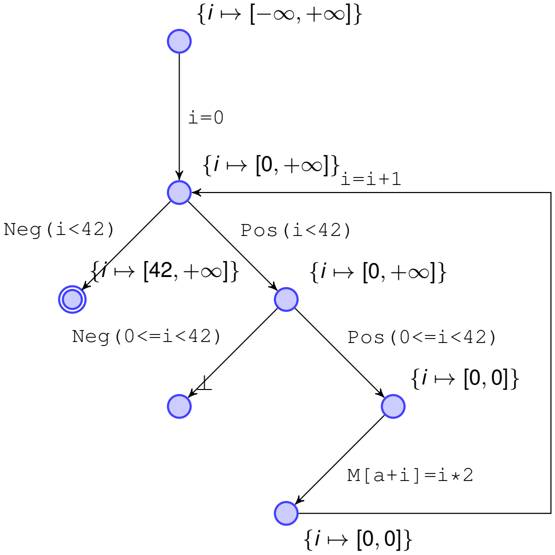

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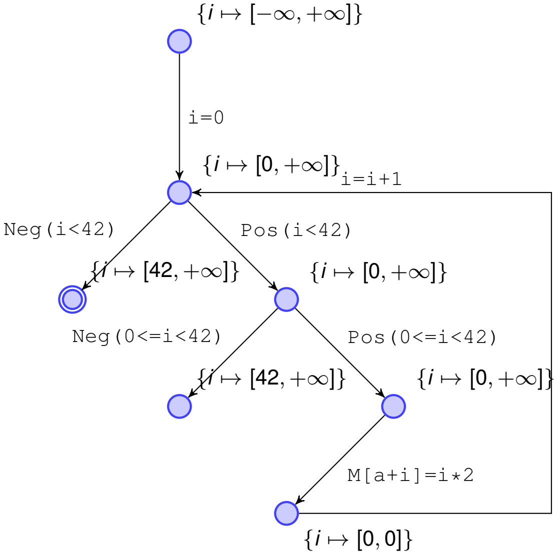

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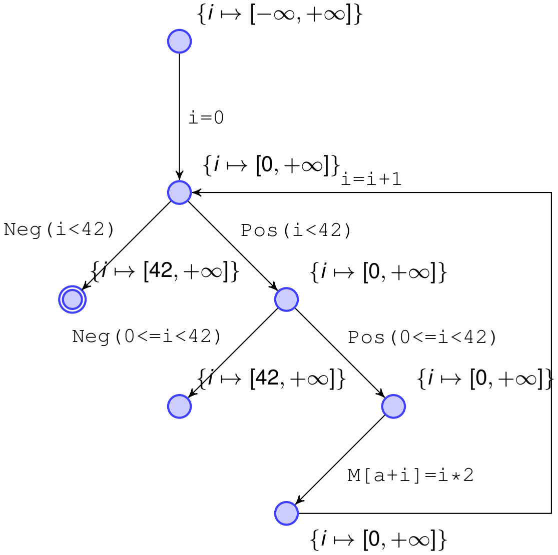

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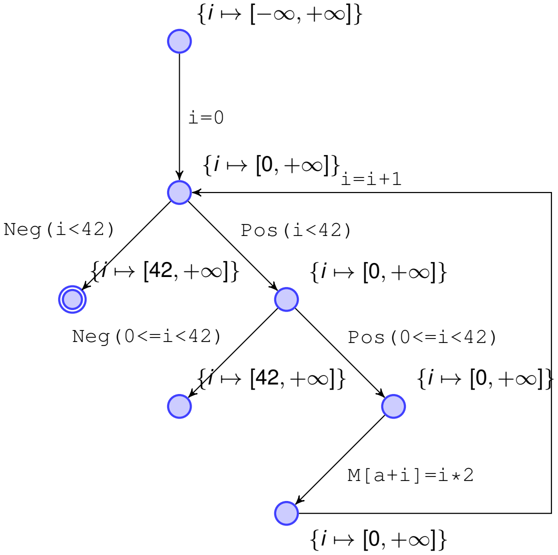


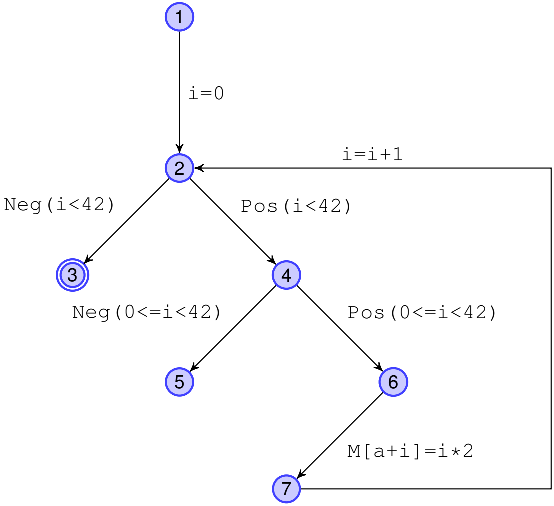
- Not exactly what we expected :(

Idea

- Only apply widening at loop separators
- A set $S \subseteq V$ is called loop separator, iff each cycle in the CFG contains a node from S.
- Intuition: Only loops can cause infinite chains of updates.
- Thus, FP-iteration still terminates

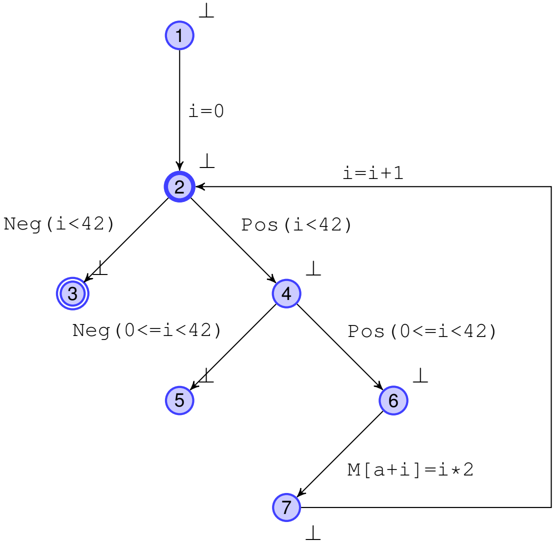
Problem

- How to find suitable loop separator



- We could take $S = \{2\}, S = \{4\}, \ldots$
- Results of FP-iteration are different!

Loop Separator $S = \{2\}$



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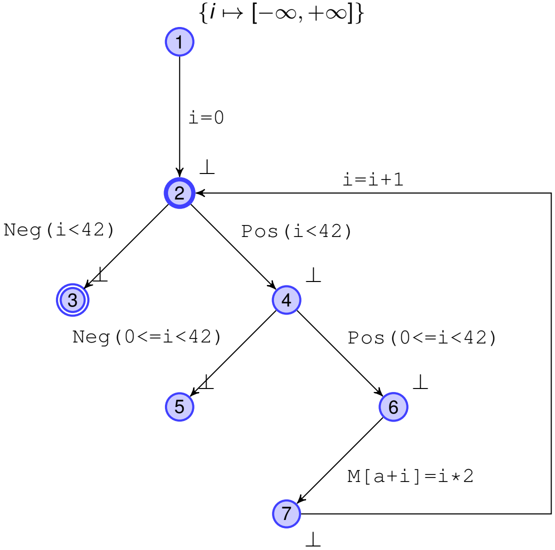

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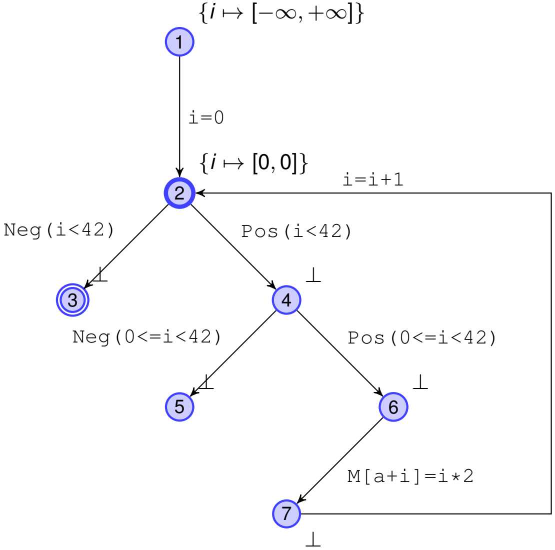

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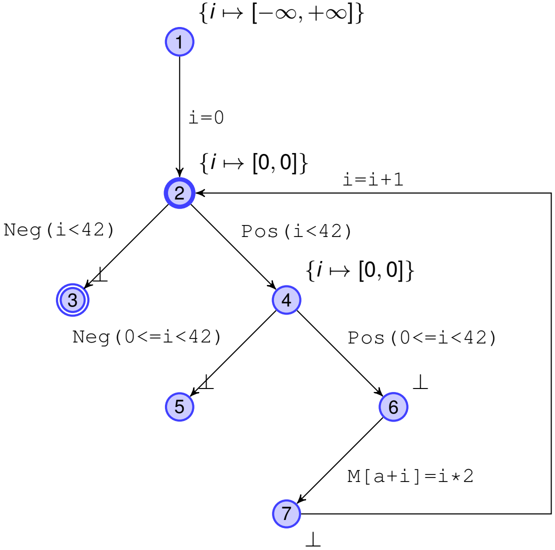

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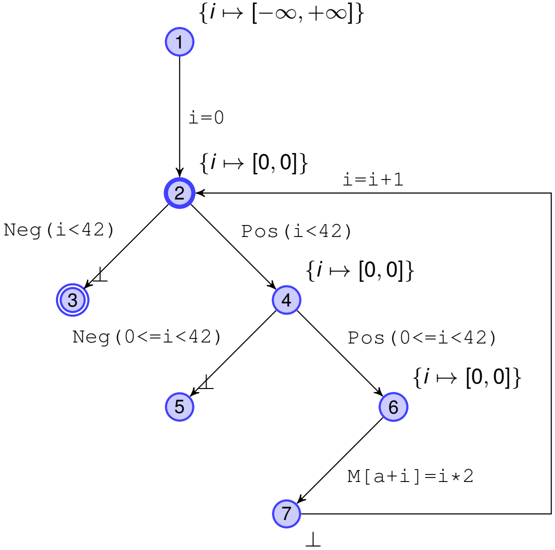

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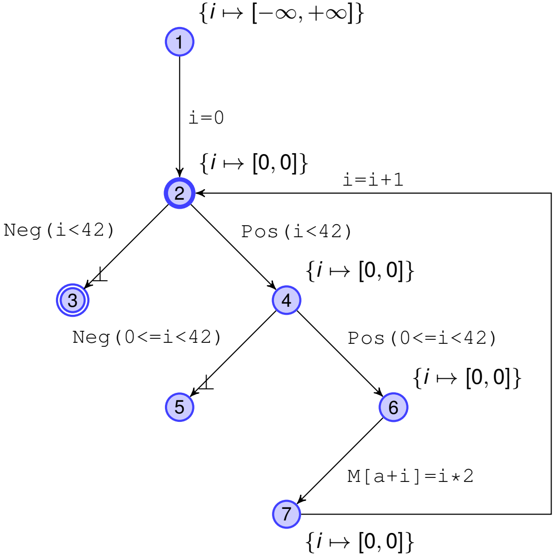

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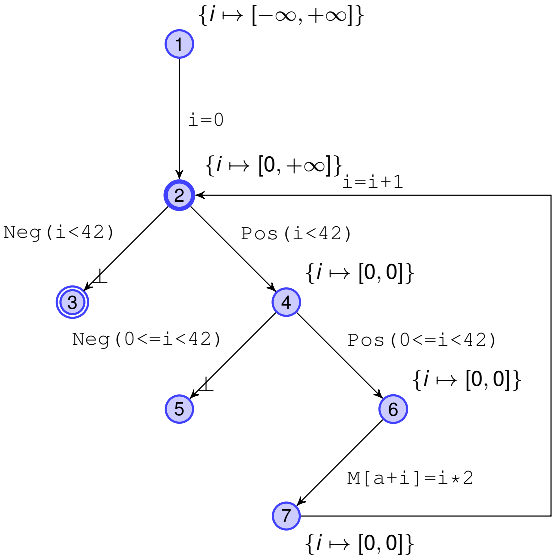

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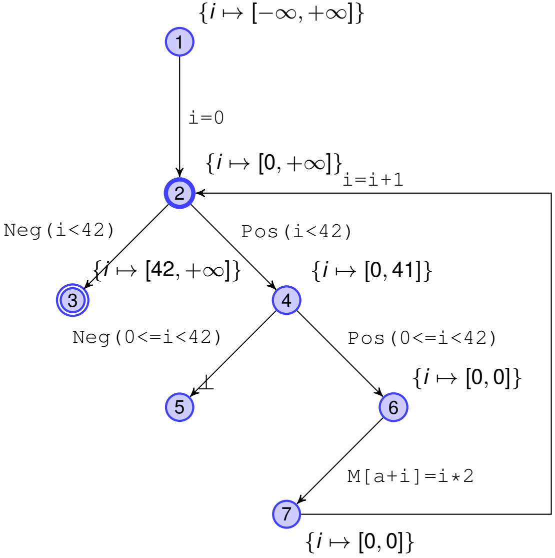

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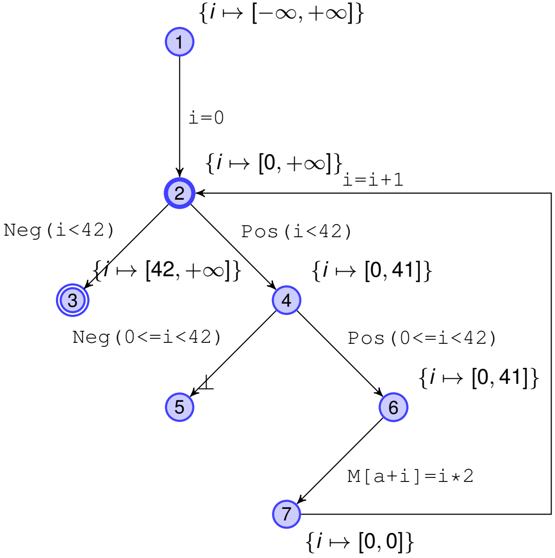

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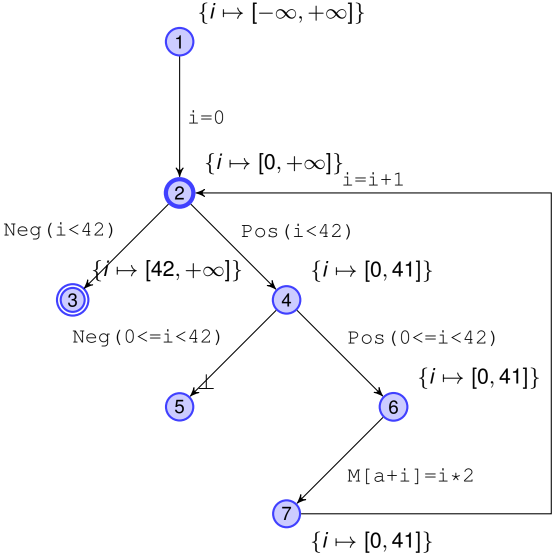

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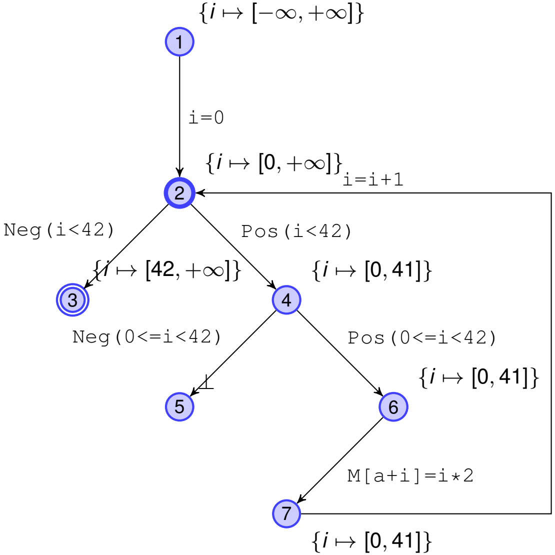
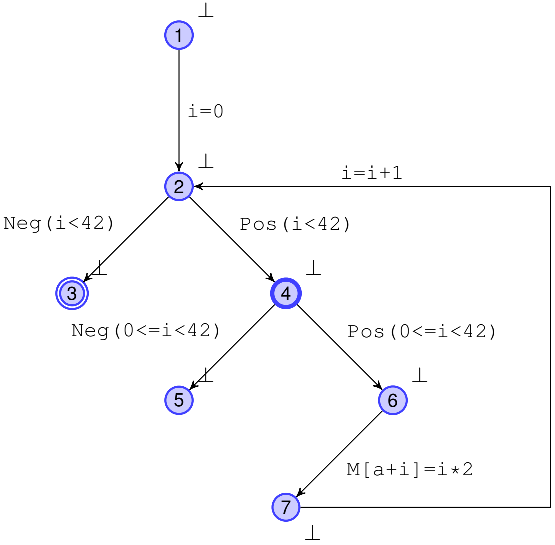


- Fixed point

Loop Separator $S = \{4\}$



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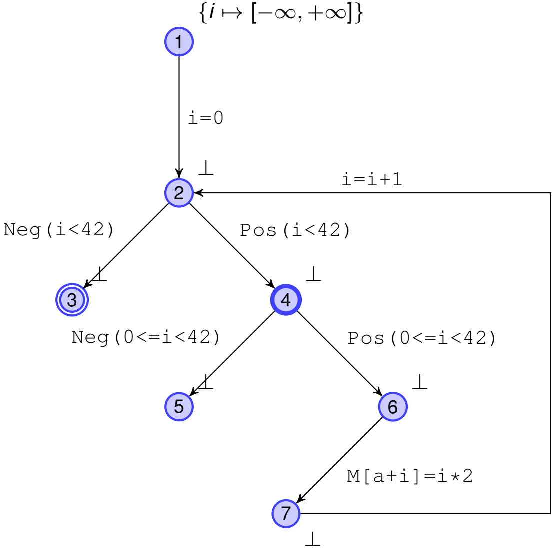

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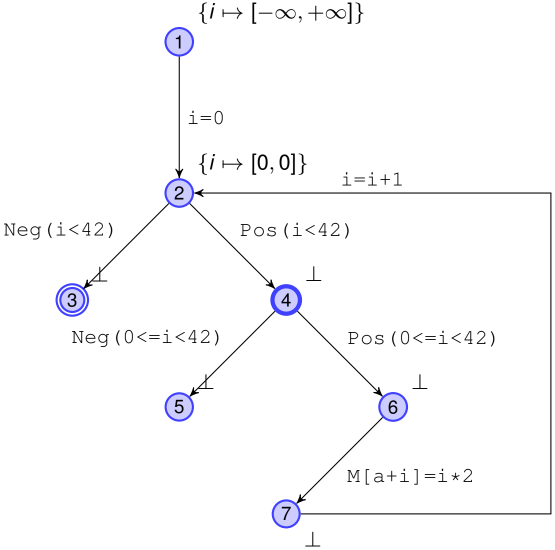

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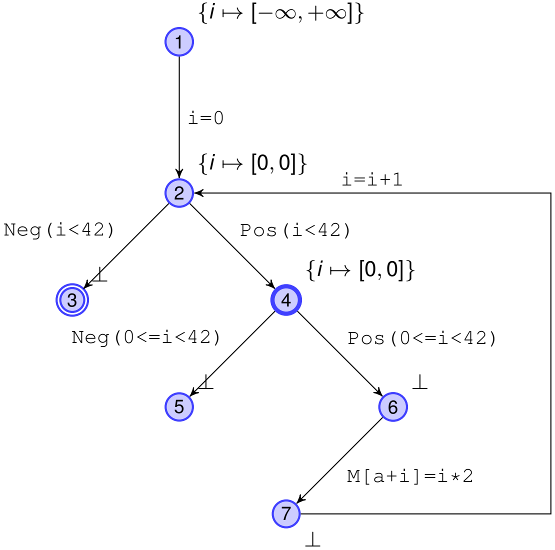

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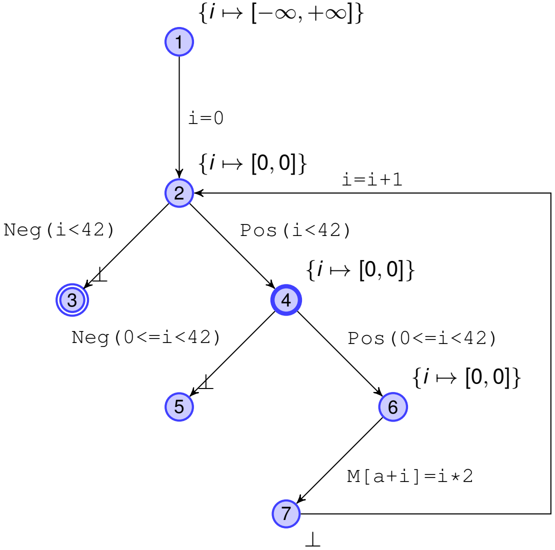

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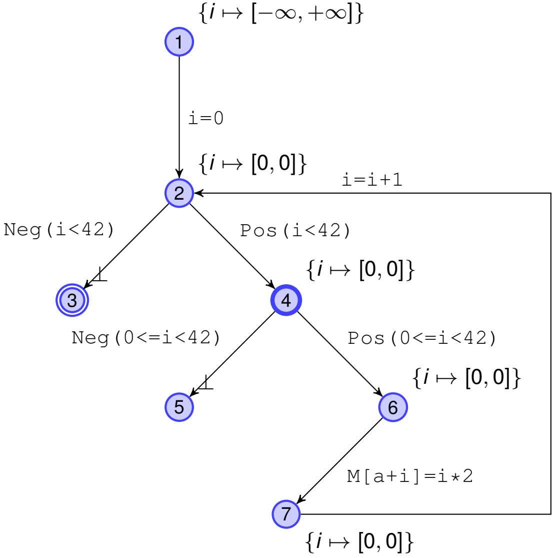

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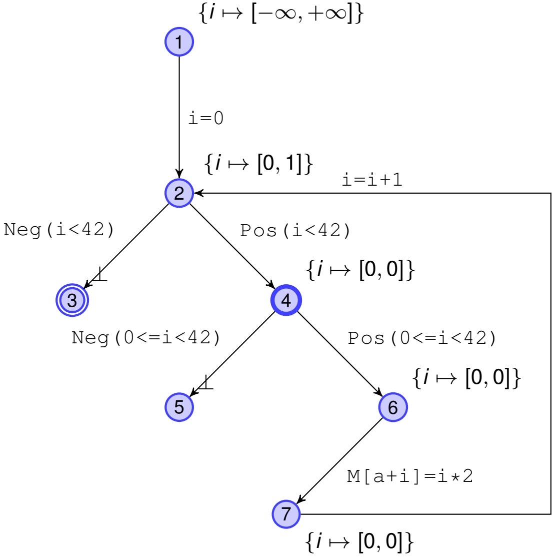

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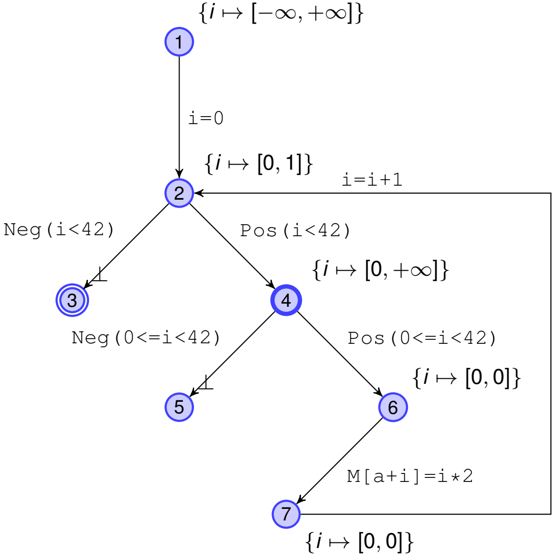

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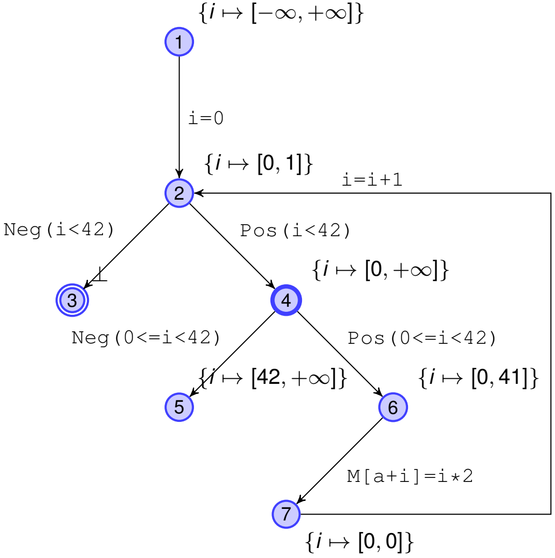

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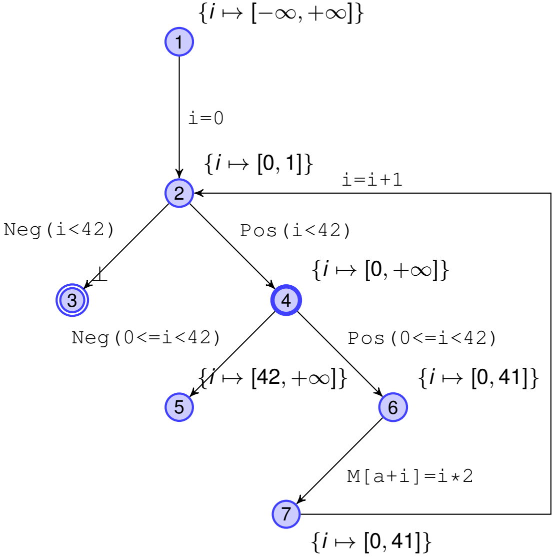

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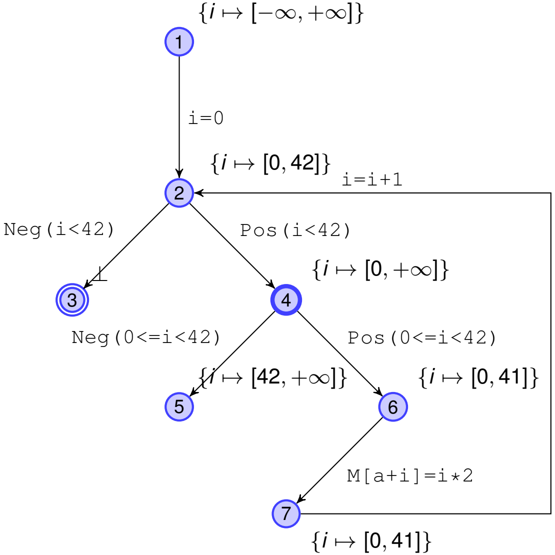

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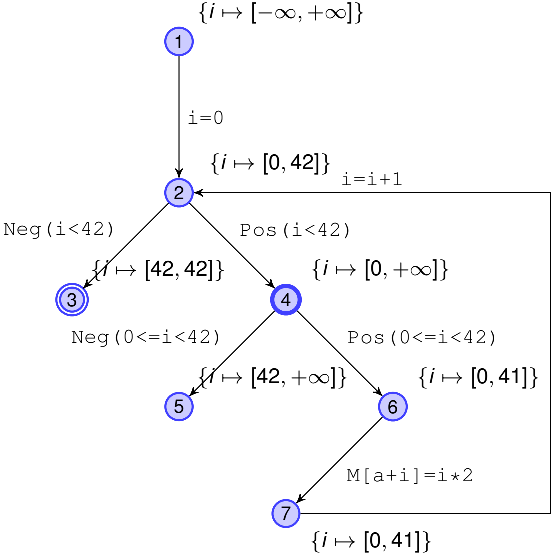

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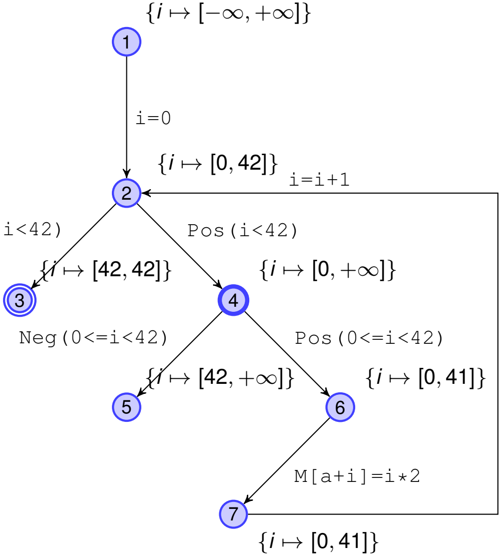

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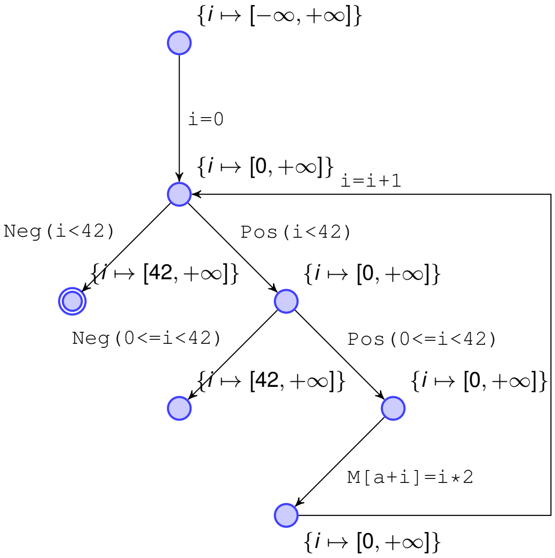

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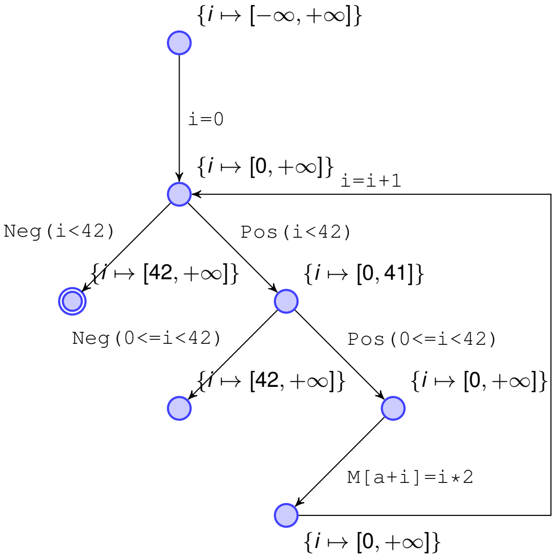

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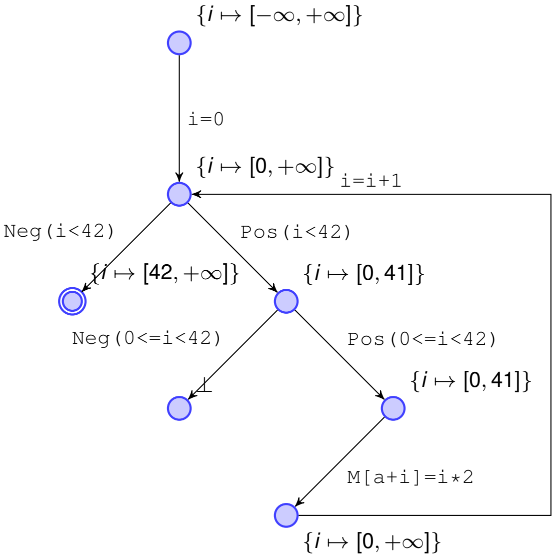

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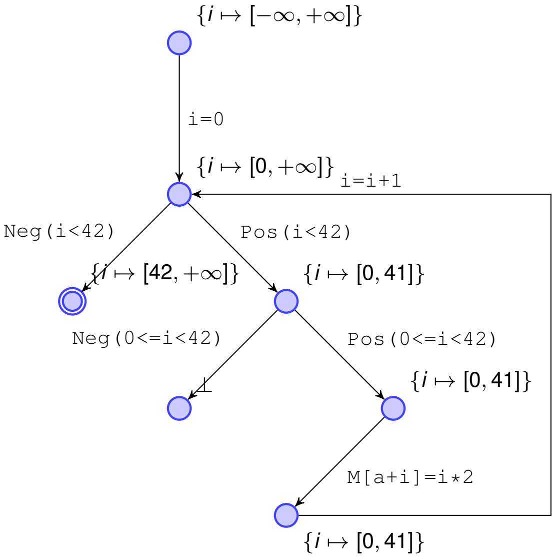

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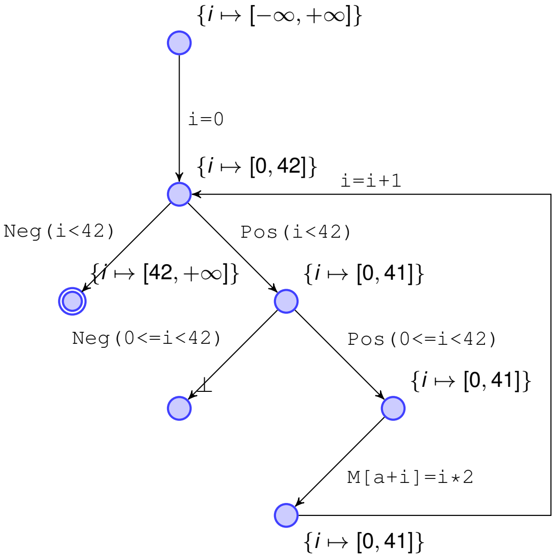

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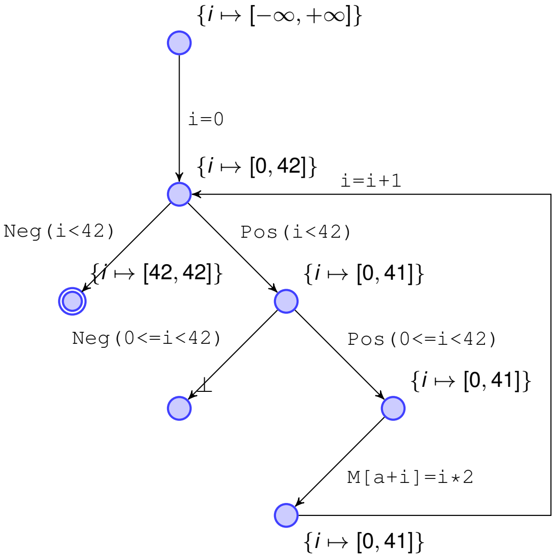

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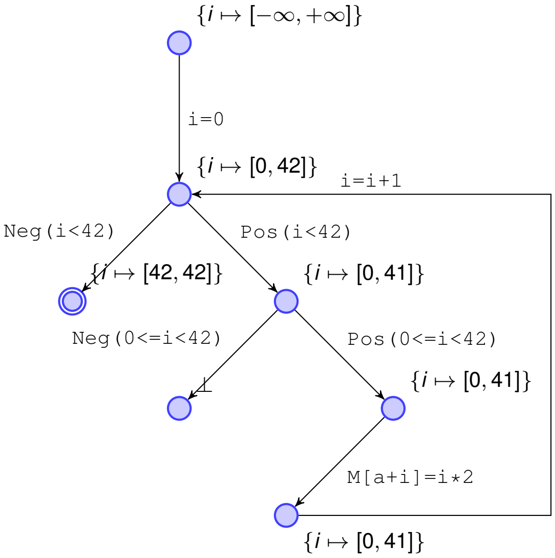

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- Question: Do we have to accept possible nontermination/large number of iterations?

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- We have: $I^k(\vec{x}) \sqsupseteq H^k(\vec{x}) = F^k(\vec{x}) \sqsupseteq F^{k+1}(\vec{x})$.
  - I.e., $I^k(\vec{x})$ greater (valid approx.) than a solution.

For interval analysis

- Preserve (finite) interval bounds: $[l_1, u_1] \sqcap [l_2, u_2] := [l, u]$, where

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- $\bar{\sqcap}$ is not commutative
- For our example: Same result as non-accelerated narrowing!

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- However: Design of good widening/narrowing remains black magic

Last Lecture

- Interval analysis (ctd)
  - Abstract values: Intervals $[l, u]$ with $l \leq u$, $l \in \mathbb{Z}_{-\infty}$, $u \in \mathbb{Z}^{+\infty}$
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- Problem: makes analysis imprecise
  - Idea 1: Widening only at loop separators
  - Idea 2: Narrowing
    - FP-Iteration on solution preserves solution
    - But may make it smaller
  - Accelerated narrowing:
  - Use narrowing operator for update, that lies "in between" $\sqcap$ and original value
    - ... and converges within finite time
    - Here: Keep finite interval bounds

Recipe: Abstract Interpretation (I)

- Define abstract value domain $\mathbb{A}$, with partial order $\sqsubseteq$
  - $\sqcup$ must be totally defined ($\sqcap$ need not always exists)
- Define description relation between values: $\Delta \subseteq \mathbb{Z} \times \mathbb{A}$
  - Show: Monotonicity: $\forall a_1 \sqsubseteq a_2, v.\ v\ \Delta\ a_1 \implies v\ \Delta\ a_2$
  - Standard: Lift to valuations $(\mathrm{Reg} \to \mathbb{A})$, domain $(\mathbb{D} := (\mathrm{Reg} \to \mathbb{A}) \cup \{\bot\})$
  - Define abstract operators $v^\# : \mathbb{A}, \square^\# : \mathbb{A} \times \mathbb{A} \to \mathbb{A}$, etc.
    - Show soundness wrt. concrete ones:

$$\forall c \in \mathbb{Z}.\ v\ \Delta\ v^\#$$

$$\forall v_1, v_2 \in \mathbb{Z},\ d_1, d_2 \in \mathbb{A}.\ v_1\ \Delta\ d_1 \wedge v_2\ \Delta\ d_2 \implies v_1 \square v_2\ \Delta\ d_1 \square^\# d_2$$

    - For free: $\rho\ \Delta\ \rho^\# \implies [\![e]\!]\rho\ \Delta\ [\![e]\!]^\# \rho^\#$
- Define transformation tr $:: \mathrm{Act} \times \mathbb{D} \to \mathrm{Act}$
  - Show correctness: $(\rho, \mu)\ \Delta\ d \implies [\![a]\!](\rho, \mu) = [\![\mathrm{tr}(a, d)]\!](\rho, \mu)$
- Define abstract effects $[\![\cdot]\!]^\# : \mathrm{Act} \to \mathbb{D} \to \mathbb{D}$, initial value $d_0 \in \mathbb{D}$
  - Usually: Creativity only required on Pos,Neg
  - Show: Monotonicity: $\forall d_1 \sqsubseteq d_2, a.\ [\![a]\!]^\# d_1 \sqsubseteq [\![a]\!]^\# d_2$ and simulation:

$$\forall \rho, \mu.\ (\rho, \mu)\ \Delta\ d_0$$

$$\forall (\rho, \mu) \in \mathrm{dom}([\![a]\!]), d.\ (\rho, \mu)\ \Delta\ d \implies [\![a]\!](\rho, \mu)\ \Delta\ [\![a]\!]^\# d$$

Recipe: Abstract Interpretation (II)

- Check finite chain height of domain
  - Finite: Done
  - Infinite (or too high)
    - Define widening, narrowing operator

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  - Constraint system. For monotonic abstract effects. Precise if distributive.

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    - Forward: Simulation; Backward: Also (kind of) simulation
  - $\mathrm{MOP}[u]$ — Abstract effects reachable at u
  - Special case: abstract interpretation — domain describes abstract values
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- Computing analysis result
  - Constraint system. For monotonic abstract effects. Precise if distributive.
  - Solving algorithms: Naive iteration, RR-iteration, worklist algorithm

Short recapture of methods so far

- Operational semantics on flowgraphs
  - Edges have effect on states. Extend to paths.
  - Collecting semantics: $[\![u]\!]$ — States reachable at u.
- Program analysis
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  - Solving algorithms: Naive iteration, RR-iteration, worklist algorithm
- Forcing convergence: Widening, Narrowing

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Table of Contents

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  - In reality: Shared variables (interrupts, threads), DMA, memory-mapped hardware, ...
  - Compilers provide, e.g., volatile annotation

First Attempt

- Available expressions:
  - Memorize loads: Load: `x = M[e]` $\mapsto$ { $T_{M[e]}$ `= M[e]; x=`$T_{M[e]}$ }
  - Effects

$$[\![T_e = e]\!]^\# A = [\![A]\!]^\# \cup \{e\} \qquad [\![T_{M[e]} = M[e]]\!]^\# A = [\![A]\!]^\# \cup \{M[e]\}$$

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- Problem: Need to be conservative on store
  - Store destroys all information about memory

Constant propagation

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- Usually, different addresses accessed at same program point
    - E.g., iterate over array
- Storing at unknown address destroys all information

Last Lecture

- Motivation to consider memory
  - Alias analysis required!
- Changing the semantics of memory
  - Pointers to start of blocks, indexing within blocks
  - No pointer arithmetic
  - Some assumptions about program correctness: Semantics undefined if
    - Program accesses address that has not been allocated
    - Indexes block out of bounds
    - Computes with addresses

Extending semantics by blocked memory

- Organize memory into blocks
  - `p = new(e)` allocates new block of size *e*
  - `x = p[e]` loads cell *e* from block *p*
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    - In reality: Type system
  - We write null and 0 synonymously

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- Assume infinite supply of addresses

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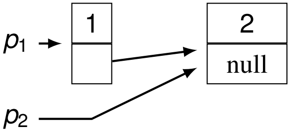
- Two states (ρ, μ) and (ρ', μ') are considered equivalent, iff they are equivalent up to permutation of addresses
  - We write $(\rho, \mu) \equiv (\rho', \mu')$
- Note: To avoid this nondeterminism in semantics:
  - Choose Addr to be totally ordered
  - Always take the smallest free address

Examples

- Building the linked list $[1, 2]$

```
p₁ = new (2)
p₂ = new (2)
p₁[0] = 1
p₁[1] = p₂
p₂[0] = 2
p₂[1] = null
```



Examples

- List reversal

```
R = null
while (T != null) {
  H = T
  T = T[0]
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  R = H
}
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- Sketch algorithm on whiteboard

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  - On store: Only destroy information for addresses that may alias with stored address

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  - On store: Only destroy information for addresses that may alias with stored address
- Must alias: Must two pointers point to the same address
  - If so, store to one can update information for the other
- Here: Focus on may-alias
  - Important to limit the destructive effect of memory updates
  - Must alias: Usually only done in local scope, by, e.g., copy propagation

First Idea

- Summarize (arbitrarily many) blocks of memory by (fixed number of) allocation sites
  - Use start node of edge in CFG to identify allocation site
  - Abstract values $\mathrm{Addr}^\# = V$, $\mathrm{Val}^\# = 2^{\mathrm{Addr}^\#}$ (Possible targets for pointer)
  - Domain: $(\mathrm{Reg} \to \mathrm{Val}^\#) \times (\mathrm{Addr}^\# \to \mathrm{Val}^\#)$
- Effects

$$
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&\dots \\
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x points to this allocation site.

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Add addresses from y to each possible address of p

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```
u: p₁ = new (2)
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p₁[0] = 1
p₁[1] = p₂
p₂[0] = 2
p₂[1] = null
```

- At end of program, we have

$$R = p_1 \mapsto \{u\}, p_2 \mapsto \{v\}$$
$$M = u \mapsto \{v\}, v \mapsto \{\}$$

Description Relation

$$(\rho, \mu) \; \Delta \; (R, M) \text{ iff } \exists s : \mathrm{Addr} \to V.\, \forall a, a' \in \mathrm{Addr}.\, \forall x, i.$$

$$\rho(x) = a \implies s(a) \in R(x) \quad (1)$$

$$\wedge\, \mu(a, i) = a' \implies s(a') \in M(s(a)) \quad (2)$$

Intuitively: There is a mapping s from addresses to allocation sites, with:

(1) If a register contains an address, its abstract value contains the corresponding allocation site

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Correctness of abstract effects (sketch)

- On whiteboard

Discussion

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  - If store is not initialized, we find out nothing
- Analysis can be quite expensive
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  - Only one instance of abstract registers/memory needed
- For our simple example: No loss in precision

First attempt

- Each edge (u, a, v) gives rise to constraints

a	constraints
$x = y$	$R(x) \supseteq R(y)$
$x = \mathrm{new}(e)$	$R(x) \supseteq \{u\}$
$x = p[e]$	$R(x) \supseteq \bigcup\{M(a) \mid a \in R(p)\}$
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Last Lecture

- Flow sensitive points-to analysis
  - Identify blocks in memory with allocation sites
  - Does not scale. One abstract memory per program point.
- Flow-insensitive points-to analysis
  - Compute one abstract memory that approximates all program points.
  - Does not scale. Too many constraints
- Flow-insensitive alias analysis
  - Compute equivalence classes of p and $p[]$

Alias analysis

- Idea: Maintain equivalence relation between variables p and memory accesses $p[]$
  - $x \sim y$ whenever x and y may contain the same address (at any two program points)

```
u: p1 = new (2)
v: p2 = new (2)
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- $\sim = \{\{p_1[], p_2\}, \{p_1\}, \{p_2[]\}\}$

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- On partitions of finite sets: Let $R = [p_1]_\sim \dot\cup \ldots \dot\cup [p_n]_\sim$
  - **`union`**`(~,p,p')`: Let $p \in [p_i]_\sim, p' \in [p_j]_\sim$
    Result: $\{[p_i]_\sim \cup [p_j]_\sim\} \dot\cup \{[p_k] \mid 1 \leq k \leq n \wedge k \notin \{i,j\}\}$

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- `union*(~,p,p')`:
  - The finest ER that is coarser than `union(~,p,p')` and satisfies (rec)

Alias analysis

```
π = { {x}, {x[]} | x ∈ Vars } // Finest ER

for (_,a,_) in E do {
  case a of
    x=y: π = union*(π,x,y)
  | x=y[e]: π = union*(π,x,y[]) // y variable
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- Start with finest ER (=)

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- Start with finest ER (=)
- Iterate over edges, and union equivalence classes

Example

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1: p₁ = new (2)
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3: p₁[0] = 1
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init	$\{\{p_1\}, \{p_2\}, \{p_1[]\}, \{p_2[]\}\}$
$1 \to 2$	$\{\{p_1\}, \{p_2\}, \{p_1[]\}, \{p_2[]\}\}$
$2 \to 3$	$\{\{p_1\}, \{p_2\}, \{p_1[]\}, \{p_2[]\}\}$
$3 \to 4$	$\{\{p_1\}, \{p_2\}, \{p_1[]\}, \{p_2[]\}\}$
$4 \to 5$	$\{\{p_1\}, \{p_2, p_1[]\}, \{p_2[]\}\}$
$5 \to 6$	$\{\{p_1\}, \{p_2, p_1[]\}, \{p_2[]\}\}$

Example

```
1: R = null
2: if Neg (T != null) goto 8
3:   H = T
4:   T = T[0]
5:   H[0] = R
6:   R = H
7: goto 2
8:
```

init	$\{\{H\},\{R\},\{T\},\{H[]\},\{T[]\}\}$
$3 \to 4$	$\{\{H,T\},\{R\},\{H[],T[]\}\}$
$4 \to 5$	$\{\{H,T,H[],T[]\},\{R\}\}$
$5 \to 6$	$\{\{H,T,H[],T[],R\}\}$
$6 \to 7$	$\{\{H,T,H[],T[],R\}\}$

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$\Longrightarrow$ Only works for whole programs, no input via memory

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  - `find(x)`: Return representative of $[x]$
  - **`union`**`(x,y)`: Join equivalence classes represented by x and y
    - Destructive update!

Union-Find: Idea

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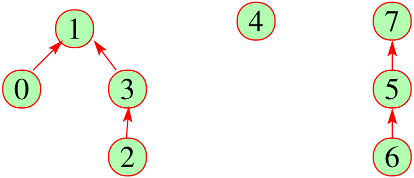

0	1	2	3	4	5	6	7
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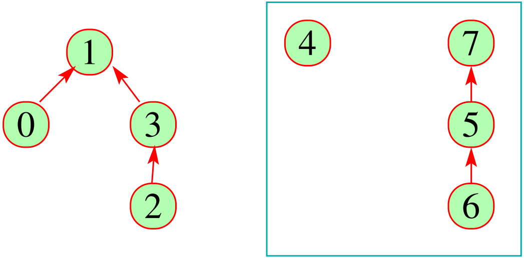

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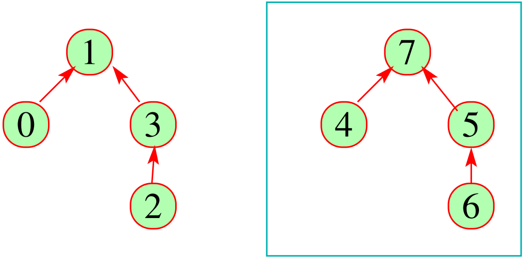

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  - Note: This complexity is optimal :)

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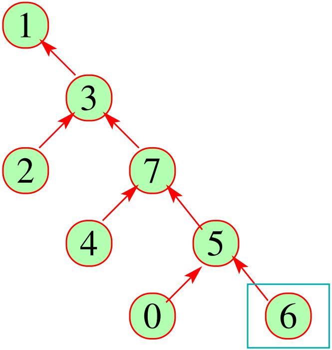

0	1	2	3	4	5	6	7
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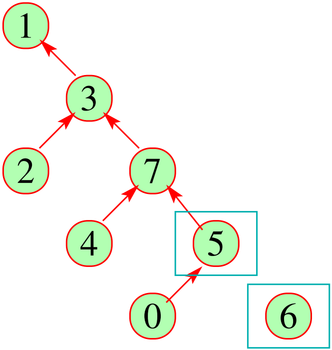

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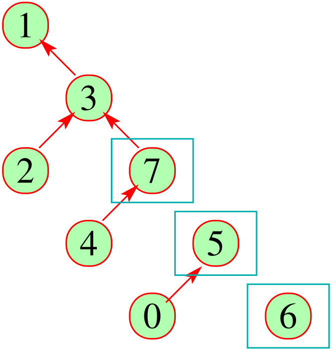

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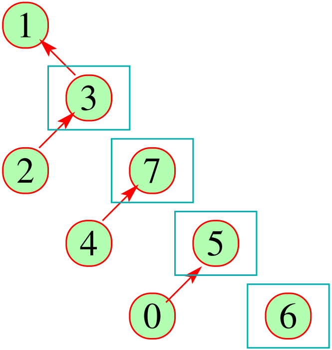

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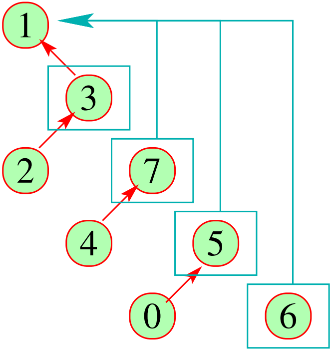

0	1	2	3	4	5	6	7
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Placing registers on top

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- Try to preserve invariant:
  - If equivalence class contains register, its representative (root node) is register
  - On union, if linking register class to non-register class:
    - Swap stored values in roots
- Then, register equivalence class can be identified by its representative

Implementing union*

```
union⋆(x,y):
  x = find(x); y=find(y)
  if x != y then
    union(x,y)
    if x ∈ Regs & y ∈ Regs then
      union⋆(x[],y[])
```

Summary

- Complexity:
  - $O(|E| + |\mathrm{Reg}|)$ calls to union*, find. $O(|\mathrm{Reg}|)$ calls to union.

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- Analysis is fast. But may be imprecise.
- More precise analysis too expensive for compilers.

Last Lecture

- Alias analysis by merging equivalence classes
- Implementation by union-find structure
  - Optimizations: Union-by-size, path-compression
  - Implementing union*

Evaluation

Please fill out evaluation forms online.

Table of Contents

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Idea

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if * {
  x = M[5]
} else {
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y₂ = x + 1
M[1]=y₁ + y₂
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- `x+1` is evaluated on every path
- On else-path even two times

Goal

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$\longmapsto$

```
if * {
  x = M[5]
  T=x + 1
} else {
  T = x + 1
  y₁ = T
}
y₂=T
M[1]=y₁ + T
```

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- Insert assignments as early as possible.
- Do not add evaluations of e that would not have been executed at all.
  - `if x!=0 then y=6 div x` $\not\mapsto$ `T=6 div x; if x!=0 then y=T`

Very busy expressions

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- e is very busy at u, if it is busy for all path from u to an end node.
- Backwards must analysis, i.e., $\sqsubseteq = \supseteq$, $\sqcup = \cap$
- Semantic intuition:
  - e busy on π — evaluation of e can be placed at start of path
  - e very busy at u — evaluation can be placed at u
    - Without inserting unwanted additional evaluations

Abstract effects

$$
\begin{aligned}
[\![\mathrm{Nop}]\!]^{\#} B &= B \\
[\![\mathrm{Pos}(e)]\!]^{\#} B &= B \cup \{e\} \\
[\![\mathrm{Neg}(e)]\!]^{\#} B &= B \cup \{e\} \\
[\![x := e]\!]^{\#} B &= (B \setminus \mathrm{Expr}_x) \cup \{e\} \\
[\![x := M[e]]\!]^{\#} B &= (B \setminus \mathrm{Expr}_x) \cup \{e\} \\
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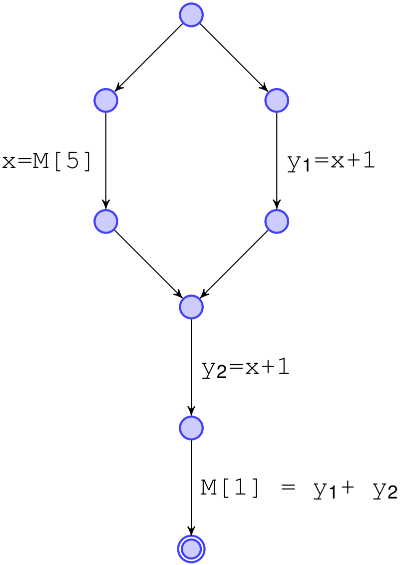
  - No very busy expressions at end nodes
- Kill/Gen analysis, i.e., distributive
  - $\mathrm{MOP} = \mathrm{MFP}$, if end node reachable from every node

Example (Very Busy Expressions)



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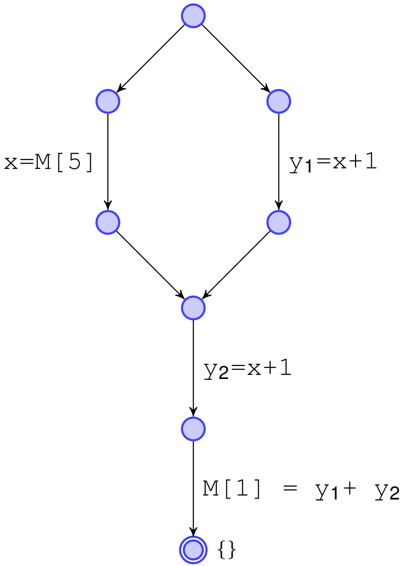

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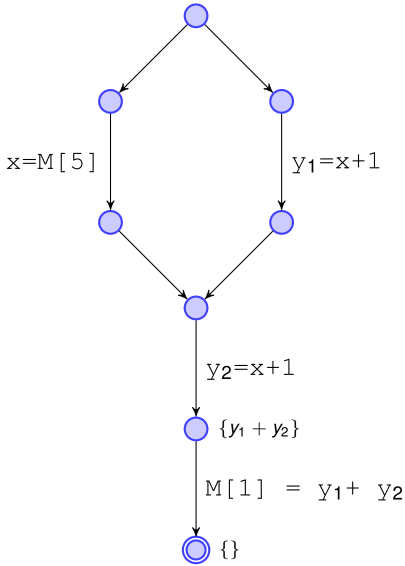

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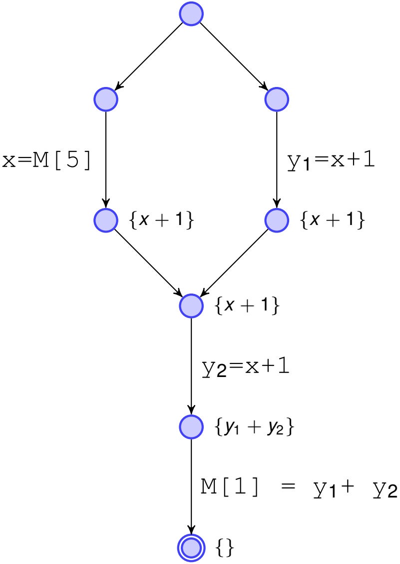

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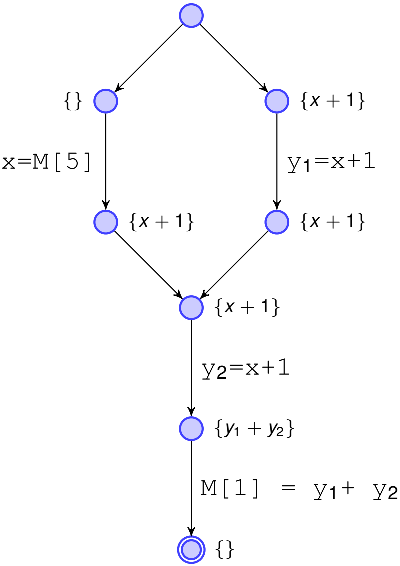

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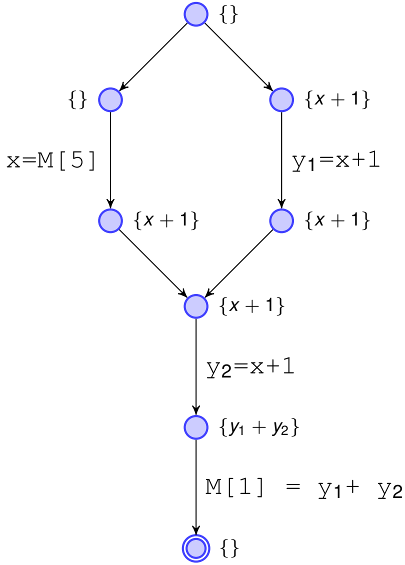

Available expressions

- Recall: Available expressions before memo-transformation

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[\![R = M[e]]\!]^{\#}_{\mathcal{A}} A &:= (A \cup \{e\}) \setminus \mathrm{Expr}_R \\
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  - Note: Order does not matter

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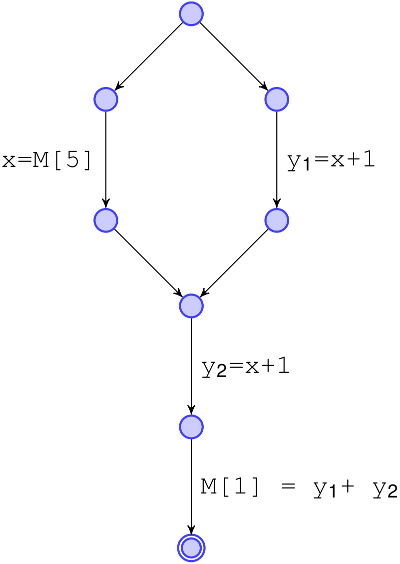
- Replace usages of expressions
  - $(u, x = e, v) \mapsto (u, x = T_e, v)$
  - analogously for other uses of e

Example

- For expression $x + 1$ only



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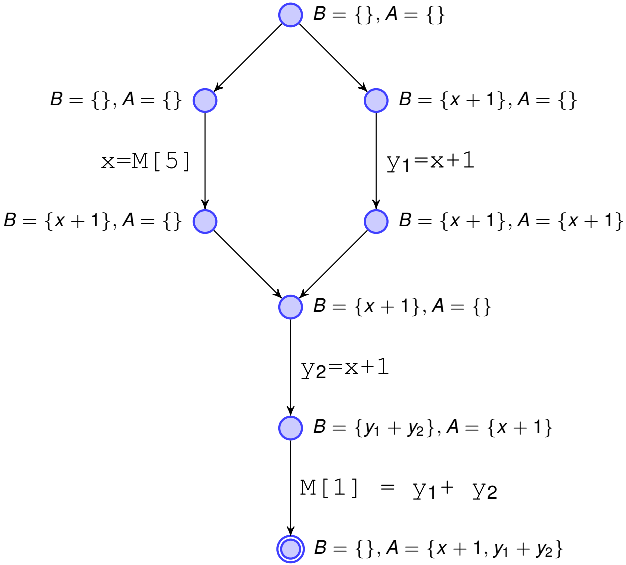

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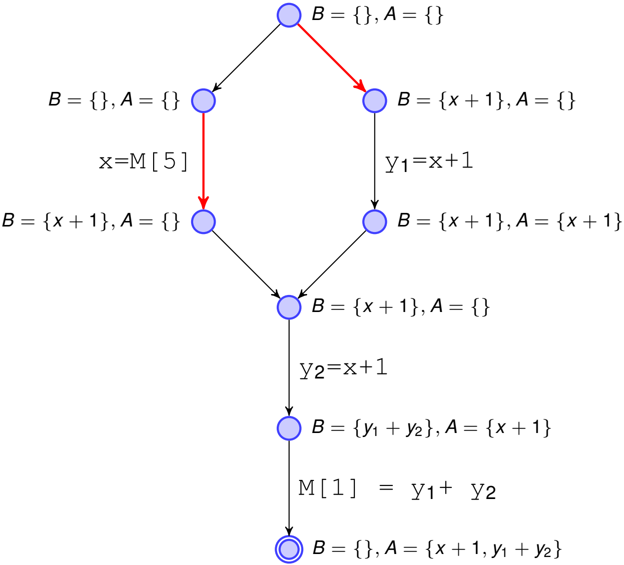

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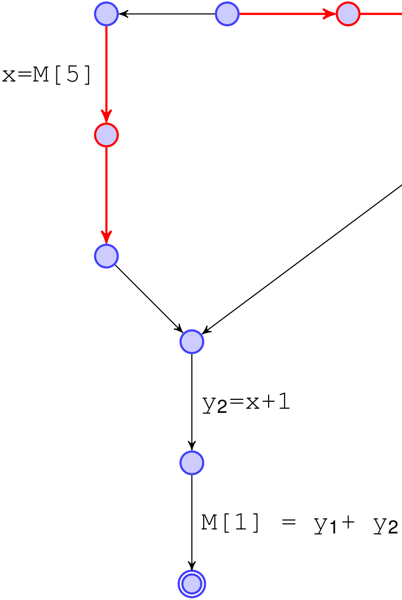

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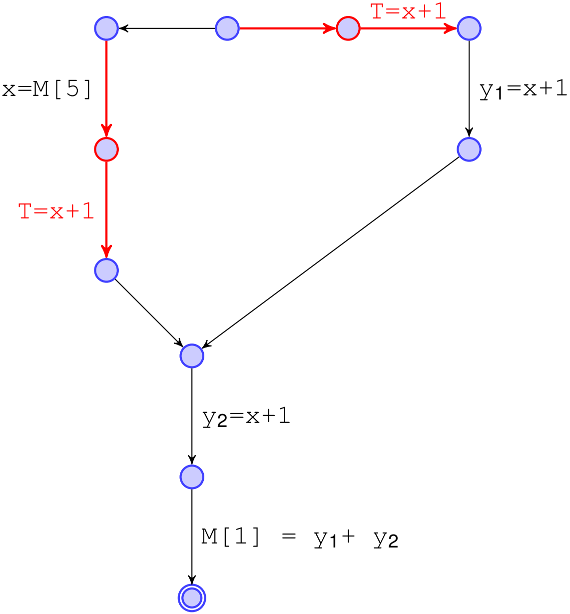

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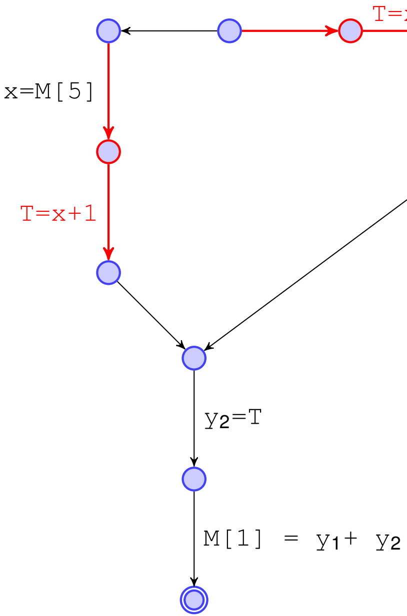

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  - Moreover, no path contains two evaluations of e, without usage of e in between
    - By contradiction. Sketch on board.

Last Lecture

- Partial Redundancy Elimination
  - Place evaluations such that
    - They are evaluated as early as possible, such that:
    - Expressions are only evaluated if also evaluated in original program
- Analysis: Very Busy Expressions
- Transformation: Placement on edges
  - where expression stops to be very busy
  - or is destroyed (and very busy at target)
- Placement only if expression is not avalable

Application: Moving loop-invariant code

```
for (i=0;i<N; ++i)
  a[i] = b + 3
```

- `b+3` evaluated in every iteration.

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- `b+3` evaluated in every iteration.
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- Should be avoided!

Example (CFG)

CFG of previous example

```
1: i=0;
2: if (i<N) {
3:    a[i] = b + 3
4:    i=i+1
5:    goto 2
6: }
```

Example (CFG)

Analysis results for expression $b + 3$

```
1: i=0;
2: if (i<N) {
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Example (CFG)

Placement happens inside loop, on edge $(2, \mathrm{Pos}(i < N), 3)$:(

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Example (CFG)

There is no node outside loop for placing *e*!

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Solution: Loop inversion

- Idea: Convert while-loop to do-while loop

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- Does not change semantics
- But creates node for placing loop invariant code

Example

CFG after loop inversion

```
1: i=0;
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- PRE may move loop-invariant code out of the loop
- Only for do-while loops
- To also cover while-loops: Apply loop-inversion first
- Loop inversion: No additional statements executed.
  - But slight increase in code size.
  - Side note: Better pipelining behavior (Less jumps executed)

Detecting loops in CFG

- Loop inversion can be done in AST

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- Idea: Predominators

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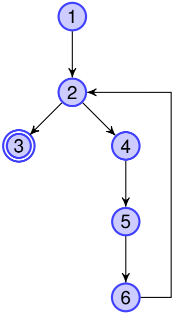
- $\Rightarrow$ is a partial order.
  - reflexive, transitive, anti-symmetric

Predominator example



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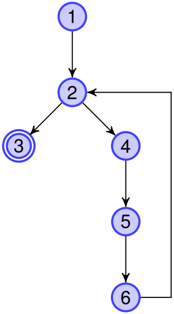

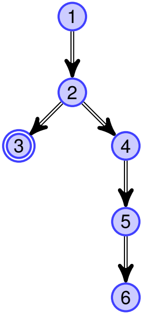

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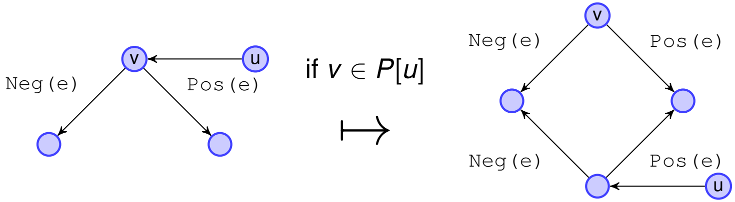
Detecting loops using predominators

- Observation: Entry node of loop predominates all nodes in loop body.



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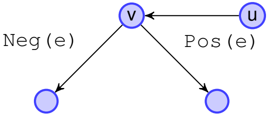

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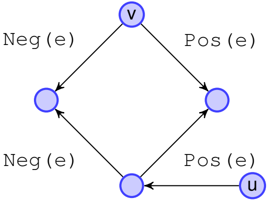

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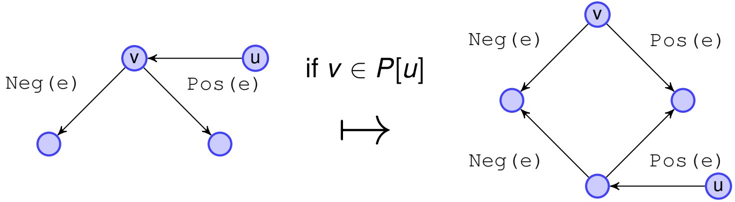

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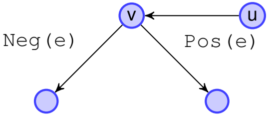

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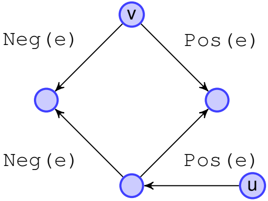
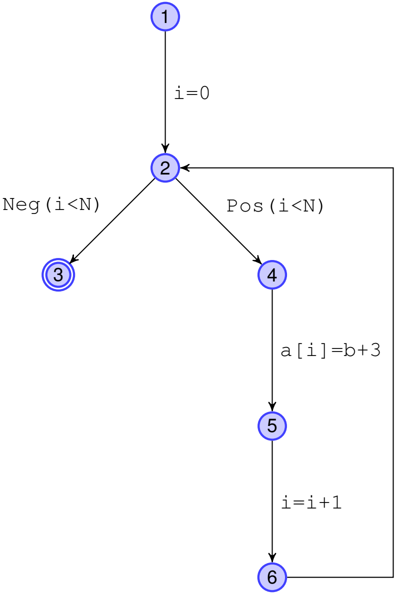


- Obviously correct

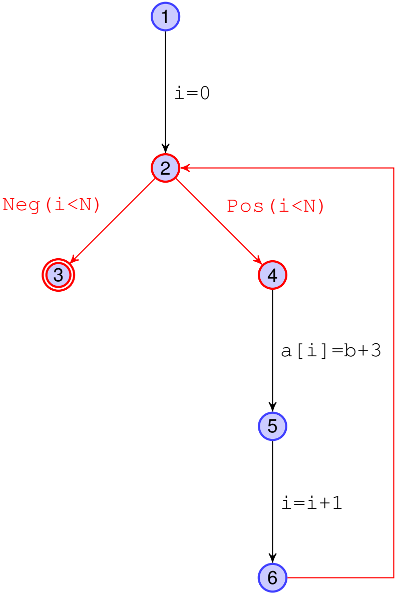
Example

CFG of running example



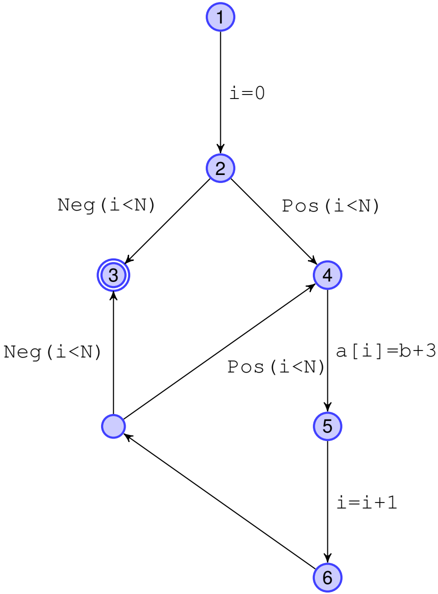
Example

$2 \in P[6]$, identified pattern for transformation



Example

Inverted loop



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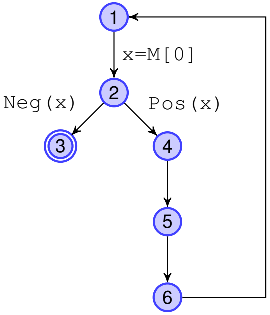

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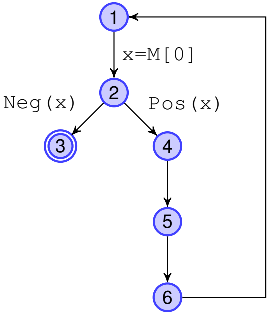

- We would have to duplicate the load-edge, too

Last Lecture

- Partial redundancy elimination
  - Very busy expressions
  - Place evaluations as early as possible
- Loop inversion
  - **`while`** $\rightarrow$ **`do-while`**
  - Enables moving loop-invariant code out of loops
  - Computation on CFG: Use pre-dominators

Table of Contents

Motivation

- Consider program

```
T = x+1
if (*) then M[0]=T
```

  - Assume (*) does not use T, and T dead at end

Motivation

- Consider program

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T = x+1
if (*) then M[0]=T
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  - Assume (*) does not use T, and T dead at end
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Motivation

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```
T = x+1
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```

  - Assume (*) does not use T, and T dead at end
- Assignment $T = x + 1$ only required on one path
- Would like to move assignment into this path

```
if (*) then {T = x+1; M[0]=T}
```

Idea

- Delay assignments as long as possible

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- Can delay assignment `x:=e` over edge k, if
  - x is not used, nor defined by k
  - No variable of e is defined by k

Delayable Assignments Analysis

- Domain: $\{x = e \mid x \in \mathrm{Reg} \wedge e \in \mathrm{Expr}\}$, Ordering: $\sqsubseteq = \supseteq$, forward
  - I.e. forward must analysis

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- Domain: $\{x = e \mid x \in \mathrm{Reg} \wedge e \in \mathrm{Expr}\}$, Ordering: $\sqsubseteq = \supseteq$, forward
  - I.e. forward must analysis
- $d_0 = \emptyset$, no delayable assignments at program start

$$
\begin{aligned}
[\![\mathrm{Nop}]\!]^{\#} D &= D \\
[\![x = e]\!]^{\#} D &= D \setminus (\mathrm{Ass}(e) \cup \mathrm{Occ}(x)) \cup \{x = e\} \\
[\![\mathrm{Pos}(e)]\!]^{\#} D &= D \setminus \mathrm{Ass}(e) \\
[\![\mathrm{Neg}(e)]\!]^{\#} D &= D \setminus \mathrm{Ass}(e) \\
[\![x = M[e]]\!]^{\#} D &= D \setminus (\mathrm{Ass}(e) \cup \mathrm{Occ}(x)) \\
[\![M[e_1] = e_2]\!]^{\#} D &= D \setminus (\mathrm{Ass}(e_1) \cup \mathrm{Ass}(e_2))
\end{aligned}
$$

where

$$\mathrm{Ass}(e) := \{x = e' \mid x \in \mathrm{Reg}(e)\} \qquad \text{Assignments to variable in } e$$

$$\mathrm{Occ}(x) := \{x' = e \mid x = x' \vee x \in \mathrm{Reg}(e)\} \qquad \text{Assignments in which } x \text{ occurs}$$

Intuition

- $x = e \in D[u]$: On every path reaching u, the assignment $x = e$ is executed, and no edge afterwards:
  - Depends on x
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- $x = e \in D[u]$: On every path reaching u, the assignment $x = e$ is executed, and no edge afterwards:
  - Depends on x
  - Changes x or a variable of e
- Thus, this assignment can be safely moved to u

Transformation

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- $(u, x = e, v) \mapsto (u, ss_1, w), (w, ss_2, v)$ where
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  - ss_2 Assignments to live variables delayable due to edge, but not at v (Other paths over v)
  - w is fresh node

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$$ss_1 := \{x' = e' \in D[u] \setminus [\![x = e]\!]^{\#} D[u] \mid x' \in L[u]\}$$

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- $v_e \in V_{\mathrm{end}} \mapsto (v_e, D[v_e], v_e')$
  - where v_e' is fresh end node, and v_e no end node any more.

Dependent actions

- Two actions a_1, a_2 are independent, iff $[\![a_1 a_2]\!] = [\![a_2 a_1]\!]$
  - Actions may be swapped

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- Two actions a_1, a_2 are independent, iff $[\![a_1 a_2]\!] = [\![a_2 a_1]\!]$
  - Actions may be swapped
- Assignments only delayed over independent actions

Correctness (Rough Sketch)

- First: $D[u]$ does never contain dependent assignments
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  - (1) No change of path
  - (2), not (3): Assignment dropped, but was dead anyway
  - (3). Three subcases: Sketch on whiteboard!
    - (3.1) $x = e$ stops being delayable due to dependent action

    $\Longrightarrow$ Assignment placed before this action, if live
    - (3.2) $x = e$ stops being delayable at node

    $\Longrightarrow$ Assignment placed after edge to this node, if live
    - (3.3) $x = e$ delayable until end

    $\Longrightarrow$ Assignment placed at end node, if live

Example

```
1: T = x+1             D: {}        L: {x}
2: if (*) then {       D: {T=x+1}   L: {T}
3:   M[0]=T            D: {T=x+1}   L: {T}
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- Placement of $T = x + 1$ before edge $(3, 4)$
  - We have $T = x + 1 \in D[3] \setminus [\![M[0] = T]\!]^{\#} D[4]$, and $T \in L[3]$

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```
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2: if (*) then {
3:   T = x+1
x:   M[0]=T
4:   Nop
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- PDE is generalization of DAE
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  - Number of assignments on each path does not increase (without proof)
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  - Assignment to dead variable will not be placed
  - As variable is dead on all paths leaving that assignment
- May also use true liveness.
- Non degradation of performance
  - Number of assignments on each path does not increase (without proof)
  - In particular: Assignments not moved into loops (Whiteboard)
- Profits from loop inversion (Whiteboard)

Conclusion

- Design of meaningful optimization is nontrivial
- Optimizations may only be useful in connection with others
- Order of optimization matters
- Some optimizations can be iterated

A meaningful ordering

LINV	Loop inversion
ALIAS	Alias analysis
AI	Constant propagation Intervals
RE CP DAE	(Simple) redundancy elimination Copy propagation Dead assignment elimination
PRE	Partial redundancy elimination
PDE	Partially dead assignment elimination

Table of Contents

Last Lecture

- Partially dead assignments
- Started semantics with procedures

Motivation

- So far:
  - Only regarded single procedure
  - But program typically has many procedures
  - Need to be pessimistic about their effect

Motivation

- So far:
  - Only regarded single procedure
  - But program typically has many procedures
  - Need to be pessimistic about their effect
- Now:
  - Analyze effects of procedures
  - Restrict to procedures without parameters/return values
  - But with local and global variables!
  - Can emulate parameters/return values!

Extending the semantics

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- Each procedure f represented by control flow graph G^f. Assume these are distinct!
- Add edge label $f()$ for call of procedure f
- Procedure main must exist

$$
\begin{aligned}
\mathrm{Conf} &= \mathrm{Stack} \times \mathrm{Globals} \times \mathrm{Store} \\
\mathrm{Globals} &= \mathrm{Glob} \to \mathrm{Val} \\
\mathrm{Store} &= \mathrm{Addr} \to \mathrm{Val} \\
\mathrm{Stack} &= \mathrm{Frame}^{+} \\
\mathrm{Frame} &= V \times \mathrm{Locals} \\
\mathrm{Locals} &= \mathrm{Loc} \to \mathrm{Val}
\end{aligned}
$$

- where Glob are global variable names, and Loc are local variable names

Execution, small-step semantics

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- Initial configuration: $([(v_0{}^{main}, \lambda x.\ 0)], \rho_g, \mu)$

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- Initial configuration: $([(v_0{}^{main}, \lambda x.\ 0)], \rho_g, \mu)$
- $\rightarrow \subseteq \mathrm{Conf} \times \mathrm{Conf}$

$$
\begin{aligned}
&((u, \rho_l)\sigma, \rho_g, \mu) \rightarrow ((v, \rho_l')\sigma, \rho_g', \mu') && \text{(basic)}\\
&\quad \text{if } (u, a, v) \in E \wedge [\![a]\!](\rho_l, \rho_g, \mu) = (\rho_l', \rho_g', \mu') &&\\
&((u, \rho_l)\sigma, \rho_g, \mu) \rightarrow ((v_0{}^f, \lambda x.\ 0)(v, \rho_l)\sigma, \rho_g, \mu) && \text{(call)}\\
&\quad \text{if } (u, f(), v) \in E &&\\
&((u, \_)\sigma, \rho_g, \mu) \rightarrow (\sigma, \rho_g, \mu) && \text{(return)}\\
&\quad \text{if } u \in V_{\mathrm{end}} \wedge \sigma \neq \varepsilon &&
\end{aligned}
$$

Example (factorial)

```
main():
  M[0] = fac(3)

fac(x):
  if (x <= 1) return 1
  else return x * fac(x-1)
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Translation to no arguments and return values

```
main():
m1:  Gx = 3;
m2:  fac()
m3:  M[0] = Gret
m4:

fac():
 f1: x = Gx
 f2: if (x <= 1) {
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A run:

$(m1, -)$
$Gx : -, Gret : -, M[0] : -$

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```

A run:

$(m2, -)$
$Gx : 3, Gret : -, M[0] : -$

Example (factorial)

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main():
m1:  Gx = 3;
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```

A run:

$(f1, x : 0)$
$(m3, -)$
$Gx : 3, Gret : -, M[0] : -$

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```

A run:

$(f2, x : 3)$
$(m3, -)$
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$(f4, x : 3)$
$(m3, -)$
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A run:

$(f5, x : 3)$
$(m3, -)$
$Gx : 2, Gret : -, M[0] : -$

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A run:

$(f2, x : 2)$
$(f6, x : 3)$
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A run:

$(f7, x : 2)$
$(f6, x : 3)$
$(m3, -)$
$Gx : 1, Gret : 2, M[0] : -$

Example (factorial)

```
main():
m1:  Gx = 3;
m2:  fac()
m3:  M[0] = Gret
m4:

fac():
 f1: x = Gx
 f2: if (x <= 1) {
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Realistic Call Semantic

- On real machine, procedure call involves
  - Save registers
  - Create stack frame
    - Push parameters, return address
    - Allocate stack space for local variables
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- Short demo: cdecl calling convention on x86

Inlining

- Procedure call is quite expensive

```
int f(int a, int b) {
  int l = a + b
  return l + l
}

int g (int a) {
  return f(a,a)
}
```

Inlining

- Procedure call is quite expensive
- Idea: Copy procedure body to call-site

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  - Exponentially!

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void m0() {x=x+1}
void m1() {m0();m0()}
void m2() {m1();m1()}
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void mN() {mN-1(); mN-1()}
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  - Inlining everything, program gets size $O(2^N)$

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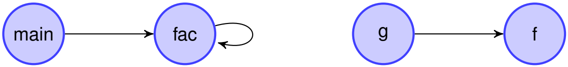

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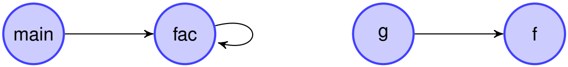

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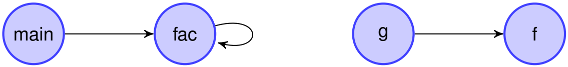

- Inline strategies
  - Leaf: Only leaf procedures
  - Everything: Every non-recursive procedure
  - Real compilers use complex heuristics
    - Based on code size, register pressure, ...

Inlining transformation

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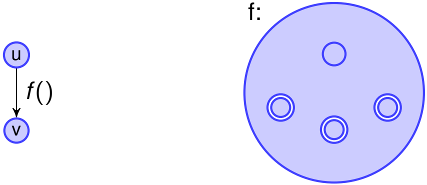

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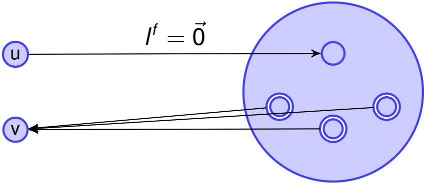

Tail call optimization

- Idea: If after recursive call, the procedure returns
- Re-use the procedure's stack frame, instead of allocating a new one

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void f() {
  if (Gi < Gn-1) {
    t = a[Gi]
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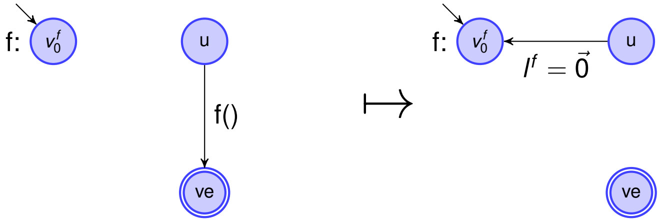
- Requires no code duplication
- Have to re-initialize local variables, according to semantics
  - Target for DAE ;)

Tail-Call Transformation



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- Crucial optimization for languages without loop construct
  - E.g., functional languages
- No duplication of code or additional local variables
- The optimization may also be profitable for non-recursive calls
  - Re-use stack-space of current frame for new stack frame
  - But not expressable in our semantics (Too high-level view on locals)

Interprocedural Analysis

- Want to extend our program analysis to procedures
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main() { int t;
  t = 0;
  if (t) M[17] = 3;
  a1 = t;
  work ();
  ret = 1 - ret;
}

work() {
  if (a1) work();
  ret = a1 ;
}
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work() {
  if (a1) work();
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}
```

$\mapsto$

```
main() { int t;
  t = 0;
  //if (t) M[17] = 3;
  a1 = 0;
  work0 ();
  ret = 1;
}

work0() {
  //if (a1) work();
  ret = 0 ;
}
```

Last Lecture

- Stack-based semantics with procedures
- Inlining optimization
- Tail-call optimization
- Path-based semantics

Generalization of Paths

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- Intuitively:
  - $f(\pi)$: Call to procedure f, which executes π and returns
  - $f_<$: Call to procedure f, which does not return
  - slpath: Same level paths, which end on same stack-level as they begin
  - Note: Inside returning call, all calls must return.

Generalization of Paths

- Recall: Paths between nodes

$$[empty]\,\frac{-}{u \xrightarrow{\varepsilon} u} \qquad\qquad [app]\,\frac{k = (u, a, v) \in E \quad v \xrightarrow{\pi} w}{u \xrightarrow{k\pi} w}$$

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- Recall

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where

$$\begin{aligned} \text{enter}(\rho_l, \rho_g, \mu) &:= (\vec{0}, \rho_g, \mu) \\ \text{combine}((\rho_l, \rho_g, \mu), (\rho_l', \rho_g', \mu')) &:= (\rho_l, \rho_g', \mu') \\ H\ e\ s &:= \text{combine}(s, (e(\text{enter}\ s))) \end{aligned}$$

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- Intuition:

enter	Set up stack frame
combine	Combine procedure result with old frame

Example

```
f () {
  if x>0 then {
    x = x - 1
    f ()
    x = x + 1
  } else {
    u: Nop
  }
}

main () {
  x = 1;
  f ()
  x = 0
}
```

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SL-path through main

```
x=1
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  f(
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)
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Path from main to u

```
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  f<
    Neg(x>0)
```

Equivalence of semantics

Theorem

The stack-based and path-based semantics are equivalent:

$$(\exists \sigma.\ ([u, \rho_l], \rho_g, \mu) \to^* ([v, \rho_l']\sigma, \rho_g', \mu')) \iff (\exists \pi.\ u \xrightarrow{\pi} v \wedge [\![\pi]\!](\rho_l, \rho_g, \mu) = (\rho_l', \rho_g', \mu'))$$

Proof sketch (Whiteboard)

- Auxiliary lemma: Same-level paths

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- Main ideas ($\impliedby$)
  - Induction on path structure
  - Executions can be repeated with stack extended at the bottom

$$(\sigma, \rho_g, \mu) \rightarrow^* (\sigma', \rho_g', \mu') \implies (\sigma\hat{\sigma}, \rho_g, \mu) \rightarrow^* (\sigma'\hat{\sigma}, \rho_g', \mu')$$

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  - Define abstract $\mathrm{enter}_f^\#$, $\mathrm{combine}_f^\#$
  - $H_f^\#\ e\ d = \mathrm{combine}_f^\#(d, e(\mathrm{enter}_f^\#(d)))$
  - $[\![f(\pi)]\!]^\# d = H_f^\#\ [\![\pi]\!]^\#\ d$
  - $[\![f_<]\!]^\# d = \mathrm{enter}_f^\#(d)$

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- Abstract effects

$$
\begin{aligned}
[\![x := c]\!]^{\#} d &= d(x := c) \quad \text{for } c \in \mathbb{Z} \\
[\![x := y]\!]^{\#} d &= d(x := d(y)) \quad \text{for } y \in \mathrm{Reg} \\
[\![x := e]\!]^{\#} d &= d(x := \top) \quad \text{for } e \in \mathrm{Expr} \setminus (\mathbb{Z} \cup \mathrm{Reg}) \\
[\![x := M(e)]\!]^{\#} d &= d(x := \top) \\
[\![\mathrm{Pos}(e)]\!]^{\#} d &= [\![\mathrm{Neg}(e)]\!]^{\#} d = [\![\mathrm{Nop}]\!]^{\#} d = [\![M(e_1) = e_2]\!]^{\#} d = d \\
\mathrm{enter}_f^{\#} d &= d(l := 0 \mid l \in \mathrm{Loc}) \\
\mathrm{combine}_f^{\#}\ d\ d' &= \lambda x.\ x \in \mathrm{Loc} ? d(x) : d'(x)
\end{aligned}
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  - Generalization of simulation proofs for intraprocedural case

Computing Solutions

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\begin{array}{rlr}
S[{v_0}^f] \sqsupseteq \mathrm{id} & & \text{(start)} \\
S[v] \sqsupseteq [\![k]\!]^\# \circ S[u] & k = (u, a, v) \in E & \text{(edge)} \\
S[v] \sqsupseteq H^\#(S[f]) \circ S[u] & k = (u, f(), v) \in E & \text{(call)} \\
S[f] \sqsupseteq S[v_e^f] & v_e^f \in V_{\mathrm{end}}^f & \text{(end)}
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- And for effects of paths reaching u

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  - But this would yield huge tables for functions
- Possible solutions:
  - Find efficient representation for functions
  - Function actually not applied to all values $d \in \mathbb{D}$. $\implies$ compute on demand.

Efficient representation of same-level effects

- Observation: Functions $S[u] \neq \bot$ are of form $\langle m \rangle$ where

$$\langle m \rangle := \lambda D\, x.\ m_1\ x \sqcup \bigsqcup_{y \in m_2\ x} D\ y$$

  - $m_1\ x : \mathbb{Z}_\bot^\top$ - Join of constants that may be assigned to x
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- Let $F := \{\langle m \rangle | m : \mathrm{Reg} \to \mathbb{Z}_\bot^\top \times 2^{\mathrm{Reg}}\}$ be the set of those functions
- To show: id, $[\![a]\!]^\# \in F$, and F closed under $\circ$, $\sqcup$, $\mathrm{enter}^\#$, and $H^\#$

Identity and effects representable

$$\text{id} = \langle \lambda x.\ (\bot, \{x\}) \rangle$$

$$[\![x := e]\!]^\# = \begin{cases} \langle \mathrm{id}(x \mapsto (c, \emptyset)) \rangle & \text{for } e = c \in \mathbb{Z} \\ \langle \mathrm{id}(x \mapsto (\bot, \{y\})) \rangle & \text{for } e = y \in \mathrm{Reg} \\ \langle \mathrm{id}(x \mapsto (\top, \emptyset)) \rangle & \text{otherwise} \end{cases}$$

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- Effects of other actions similarly

Closed under function composition and join

$$\langle m \rangle \circ \langle m' \rangle = \langle \lambda x.\ (m_1\ x \sqcup \bigsqcup_{y \in m_2\ x} m'_1\ y,\ \bigcup_{y \in m_2\ x} m'_2\ y) \rangle$$

$$\langle m \rangle \sqcup \langle m' \rangle = \langle m \sqcup m' \rangle$$

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$$\langle m \rangle \circ \langle m' \rangle = \langle \lambda x.\ (m_1\ x \sqcup \bigsqcup_{y \in m_2\ x} m_1'\ y, \bigcup_{y \in m_2\ x} m_2'\ y) \rangle$$

$$\langle m \rangle \sqcup \langle m' \rangle = \langle m \sqcup m' \rangle$$

- Intuition: Assigned constants by m_1, or by m_1', and variable goes through m_2
  - $[\![x := c; foo]\!]^\#$, or $[\![x := y; y := c]\!]^\#$
  - Note: If x not touched, we have $m_2\ x = \{x\}$
- Note: $\sqcup$ defined pointwise: $(m \sqcup m')\ x = (m_1\ x \sqcup m_1'\ x, m_2\ x \cup m_2'\ x)$

Closed under $\mathrm{enter}^{\#}$ and $H^{\#}$

$$\begin{aligned}
\mathrm{enter}^{\#} &= \langle(\lambda x.\ (0, \emptyset))|_{\mathrm{Loc}}\rangle \oplus \mathrm{id}|_{\mathrm{Glob}} \\
H^{\#}(\langle m\rangle) &= \mathrm{id}|_{\mathrm{Loc}} \oplus (\langle m\rangle \circ \mathrm{enter}^{\#})|_{\mathrm{Glob}} \\
\langle m\rangle|_{\mathrm{Loc}} \oplus \langle m'\rangle|_{\mathrm{Glob}} &:= \langle \lambda x.\ x \in \mathrm{Loc} ? m\ x : m'\ x\rangle
\end{aligned}$$

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$$\langle m\rangle|_{\mathrm{Loc}} \oplus \langle m'\rangle|_{\mathrm{Glob}} := \langle\lambda x.\ x \in \mathrm{Loc} ? m\ x : m'\ x\rangle$$

- Intuition
  - Function call only affects globals
  - $\mathrm{enter}^{\#}$ is effect of entering function (set locals to 0)
  - $f_{\mathrm{Loc}} \oplus f'_{\mathrm{Glob}}$ - Use f for local variables, f' for global variables

Recall initial example

```
main() { int t;
  t = 0;              // t=0, a1=⊤, ret=⊤
  if (t)              // t=0, a1=⊤, ret=⊤
    M[17] = 3;        // t=0, a1=⊤, ret=⊤
  a1 = t;             // t=0, a1=⊤, ret=⊤
  work ();            // t=0, a1=0, ret=⊤
  ret = 1 - ret;      // t=0, a1=0, ret=0
}                     // t=0, a1=0, ret=⊤

work() {
  if (a1) {        // id                                a1=0, ret=⊤
    work()         // id                                a1=0, ret=⊤
    Nop }          // id[ ret->(⊥,{a1}) ]            a1=0, ret=0
  ret = a1 ;       // id[ ret->(⊥,{ret,a1}) ]  a1=0, ret=⊤
}                  // id[ ret->(⊥,{a1}) ]            a1=0, ret=0
```

Discussion

- At least copy-constants can be determined interprocedurally
- For that, we had to ignore conditions and complex assignments
- However, for the reaching paths, we could have been more precise
- Extra abstractions were required as
  1. Set of abstract same-level effects must be finite
  2. and efficiently implementable

Last Lecture

- Copy-Constant propagation
- Functional approach to interprocedural analysis
  - Compute same-level effects by constraint system
  - Find efficient representation for same-level effects

Idea: Evaluation on demand

- Procedures often called only for few distinct abstract arguments
  - Observed early (Sharir/Pneuli'81, Cousot'77)
- Only analyze procedures for these
- Intuition: $[\![f, a]\!]^\#$ - effect of f if called in abstract state a
- Put up constraint system

$$
\begin{array}{rcll}
[\![{v_0}^f, a]\!]^\# & \sqsupseteq & a & \\
[\![v, a]\!]^\# & \sqsupseteq & [\![k]\!]^\#([\![u, a]\!]^\#) & \text{for basic edge } k = (u, -, v) \\
[\![v, a]\!]^\# & \sqsupseteq & \mathrm{combine}^\#([\![u, a]\!]^\#, [\![g, \mathrm{enter}^\#([\![u, a]\!]^\#)]\!]^\#) & \text{for call edge } k = (u, g(), v) \\
[\![f, a]\!]^\# & \sqsupseteq & [\![v_e^f, a]\!]^\# & \text{for } v_e^f \in V_{\mathrm{end}}^f
\end{array}
$$

- Idea: Keep track of effect for any node of procedure

Evaluation on demand

- This constraint system may be huge
- Idea: Only evaluate $[\![f, a]\!]^{\#}$ for values a that actually occur
  - Local fixed-point algorithms (not covered)
- But, we can do an example nevertheless :)

Example: Full constant propagation

```
// a1,ret | locals
main() { int t;
  t = 0;              ⊤,⊤ | 0
  if (t)              ⊤,⊤ | 0
    M[17] = 3;        ⊥
  a1 = t;             ⊤,⊤ | 0
  work ();            0,⊤ | 0
  ret = 1 - ret;      0,0 | 0
}                     0,1 | 0

work() {              [[work,(0,⊤)]]#
  if (a1)             0,⊤
    work()            ⊥
  ret = a1 ;          0,⊤
}                     0,0
```

- Only need to keep track of a_1 for calling context of `work`

Discussion

- This analysis terminates, if

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- This analysis terminates, if
  - $\mathbb{D}$ has finite height,
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- Analogous algorithms have proved efficient for analysis of PROLOG
- Together with points-to analysis, algorithms of this kind used in the Goblint-Tool
  - Data-race detection for C with POSIX-Threads

Crude approximation

- Start with very crude approximation:

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- I.e, for $(u, f(), v)$, generate constraints

$$D[{v_0}^f] \sqsupseteq \mathrm{enter}_f^\# D[u]$$

$$D[v] \sqsupseteq \mathrm{combine}_f^\#(D[u], D[v_e^f]) \qquad v_e^f \in V_{\mathrm{end}}^f$$

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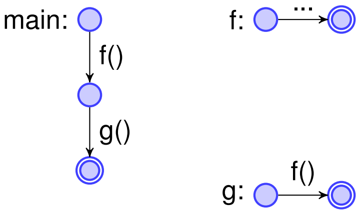
- Clearly covers all possible paths
- But also infeasible ones

Crude approximation, example

```
f () {...}
g () { f() }

main () {
  f ();
  g ()
}
```



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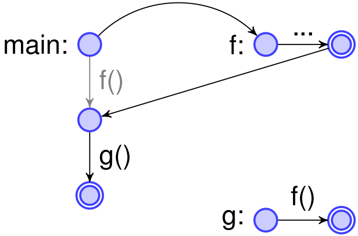

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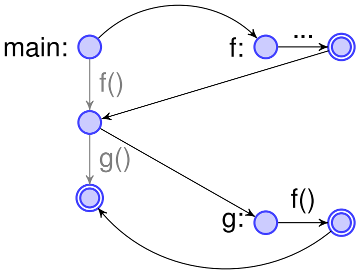

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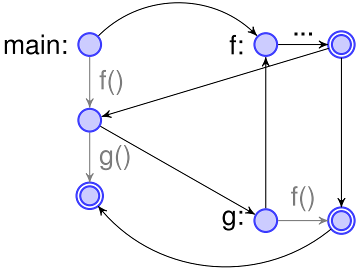

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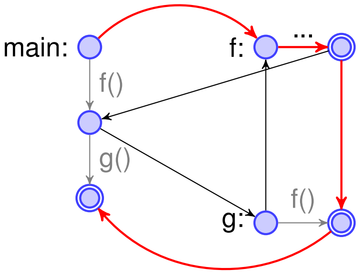

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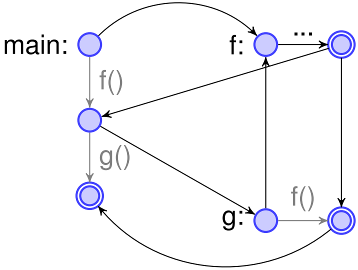

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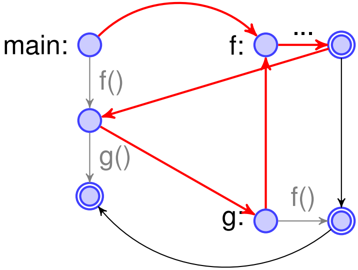

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Call strings

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- These are the topmost k return addresses on the stack
- Analyze procedures for every (feasible) call-string
- Only create edges that match call-string

Call strings

$$
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D[{v_0}^f, (v\omega)|_k] & \sqsupseteq \text{enter}^\#(D[u,\omega]) & (u, f(), v) \in E \\
D[v,\omega] & \sqsupseteq \text{combine}^\#(D[u,\omega], D[f, (v\omega)|_k]) & (u, f(), v) \in E \\
D[f,\omega] & \sqsupseteq D[v_e,\omega] & v_e \in V^f_{\text{end}}
\end{array}
$$

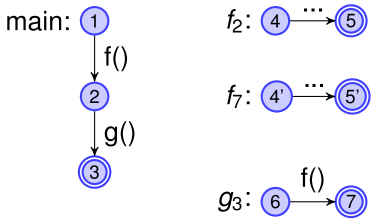
$$
\begin{array}{rll}
D[{v_0}^{\text{main}}, \varepsilon] & \sqsupseteq d_0 & \\
D[v,\omega] & \sqsupseteq [\![k]\!]^\# D[u,\omega] & k = (u, a, v) \in E
\end{array}
$$

- where $((\cdot)|_k)$ limits string size to k, cutting off nodes from the end

Example

```
f () {...}
g () { f() }

main () {
  f ();
  g ()
}
```



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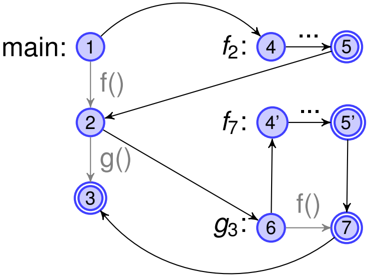

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Data races

- Concurrent accesses to global data, one is a write

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int g = 0;
t1 () {
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- Depends on machine, other programs, OS, start time, ...

Locks

- Threads can acquire/release locks

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int g = 0; lock lg;
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Demo: Goblint data race analyzer

- Program with data race
- Try to show bad reproducibility + dependence on machine load, etc.
- Show goblint-analyzer to find the race

`http://goblint.in.tum.de`

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Parallel flowgraphs with fork

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- Note: We assume that a thread only releases locks that it possesses.
- We assume that a thread does not acquire a lock it already possesses.
- Invariant: For each reachable state, the thread's lock-sets are disjoint

$$\{(v_0,\emptyset)\} \to^* \{(u_1,L_1),(u_2,L_2)\} \,\dot{\cup}\, s \implies L_1 \cap L_2 = \emptyset$$

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  - Check whether they may happen in parallel

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(*R.node*) Interesting node reachable from itself

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(*MHP.edge*) If this edge executed, other threads still at same positions

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(*MHP.trans*) Start node of forked thread parallel to other threads

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R[u] \supseteq \{u\} & \text{if } u \text{ interesting} & \text{(R.node)} \\
R[u] \supseteq R[v] & \text{if } (u, \_, v) \in E & \text{(R.edge)} \\
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$$

($MHP.fork1$) Forking thread parallel to everything that may be reached from forked thread

Lock-insensitive MHP

- Put up constraint system, $R[u]$: Set of (interesting) nodes reachable from u
  - Reachable also over forks

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$$

($MHP.fork2$) Forked thread parallel to everything that may be reached from forking thread

Correctness

- For interesting nodes u and v (also u=v), we have:

$$\exists s.\ \{v_0\} \to^* \{u, v\} \,\dot\cup\, s \implies u \in \mathrm{MHP}[v]$$

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  - Find the crucial fork, where u is reached from, wlog, the forked thread, and v is reached from the forking thread
    - $\{v_0\} \to^* \{a\} \dot{\cup} \ldots$, and $(a, \mathrm{fork}(c), b) \in E$, and $\{b\} \to^* \{u\} \dot{\cup} \ldots$, and $\{c\} \to^* \{v\} \dot{\cup} \ldots$

Lock-set analysis

- Forward, must analysis (standard)

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\begin{array}{rcll}
LS[v_0] & \subseteq & \emptyset & \\
LS[w] & \subseteq & \emptyset & (u, \mathrm{fork}(w), v) \in E \\
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- Correctness:

$$l \in LS[u] \implies (\forall s.\ \{(v_0, \emptyset)\} \to^* \{(u, L)\} \mathbin{\dot\cup} s \implies l \in L)$$

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- If satisfied, report „definitely no data race”
- Otherwise, report possible data race

Example

```
     int g = 0; lock lg;
     t1 () {
1:     acquire(lg);          R: 2      MHP: {7,11} L: {}
2:     g = g + 1;            R: 2      MHP: {7,11} L: {lg}
3:     release(lg);          R: {}     MHP: {7,11} L: {lg}
4:   }                                 MHP: {7,11} L: {}
     main () {
5:     fork t1;              R: 2,7,11 MHP: {}     L: {}
6:     acquire(lg);          R: 7,11   MHP: {2}    L: {}
7:     g = g + 1;            R: 7,11   MHP: {2}    L: {lg}
8:     release(lg);          R: 11     MHP: {2}    L: {lg}
9:     join;                 R: 11     MHP: {2}    L: {}
10:    acquire(lg);          R: 11     MHP: {2}    L: {}
11:    print g               R: 11     MHP: {2}    L: {lg}
12:    release(lg);          R: {}     MHP: {2}    L: {lg}
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```

- Check lock-sets for 2/7 and 2/11

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- Check lock-sets for 2/7 and 2/11
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- Program is safe!

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- Freedom of data races often not enough

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int x[N];
void norm() {
  lock l; n = length(x); unlock l;
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- Advanced locking patterns

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- Advanced locking patterns
  - E.g., lock chains:

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lock 1; lock 2; unlock 1; lock 3; unlock 2 ...
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$\Longrightarrow$ Transactionality

- Advanced locking patterns
  - E.g., lock chains:

    ```
    lock 1; lock 2; unlock 1; lock 3; unlock 2 ...
    ```

  - Two lock-chains executed simultaneously will never overtake

Last Lecture

- Analysis of parallel programs
  - Intraprocedural with thread creation
  - May-happen in parallel + lockset analysis = datarace analysis
- Caveats
  - Need to abstract program into model with fixed locks
    - Problematic if locks are addressed via pointers/arrays
  - Datarace freedom may no be enough
    - Transactions
    - Advanced locking patterns like lockchains

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Motivating Example

```
for (i=l;i<r;i=i+h) {
  a=a0 + b*i
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- Element size of array: b
- Loop requires $r - l$ multiplications
- Multiplications are expensive, addition much cheaper
- Observation: From one iteration of the loop to the next:
  - Difference between as is constant: $(a_0 + b(i + h)) - (a_0 + bi) = bh$

Optimization

- First, loop inversion

```
i=l;
if (i<r) {
  do {
    a=a0 + b*i
    M[a] = ...
    i=i+h
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Optimization

- First, loop inversion
- Second, pre-compute difference and replace computation of *a*
  - No multiplication left in loop

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i=l;
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  - Get rid of *i* altogether

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if (l<r) {
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  do {
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    a=a+delta
  } while (a<N)
}
```

In general

- Identify
  - loops
  - iteration variables
  - constants
  - Matching use structures

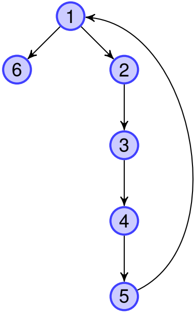
Loops

- Identify loop by node v where back-edge leads to, i.e., $(u, a, v) \in E$ with $v \Rightarrow u$
- Nodes of loop:

$$\mathrm{loop}[v] = \{w \mid w \to^* v \land v \Rightarrow w\}$$

  - I.e., nodes which can only be reached via v, and from which v can be reached again

Example



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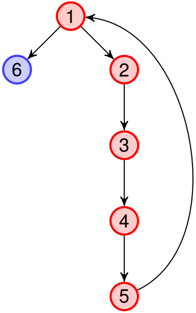

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- Replace expression $a_0 + b * i$ by a

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  - For all edges (u, a, v), we have

$$(\rho, \mu) \in I_u \cap \mathrm{dom}([\![a]\!]) \implies [\![a]\!](\rho, \mu) \in I_v$$

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    - Other edges: Do not modify variables in invariant

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\end{aligned}
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Sub-Problem: Elimination of Nop

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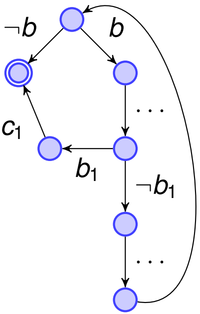
Motivation

- Translate CFG to instruction list
- Need to insert jumps. No unique translation.
- Crucial for performance

```
while (b) {
  ...
  if (b1) {
    c1;
    break;
  }
  ...
}
```



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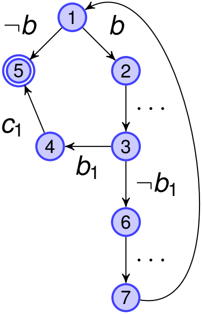


```
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     jneg b1 6
     c1
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6:   ...
     jmp 1
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Bad linearization, jump in loop

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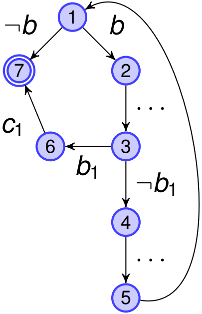


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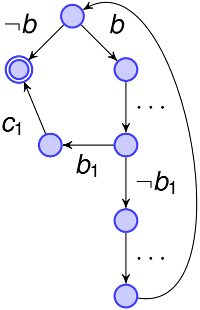
2. Linearize
   - Pre-order DFS to number nodes
   - Visit hotter successors first

Example



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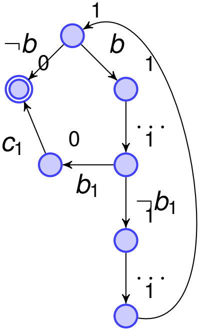

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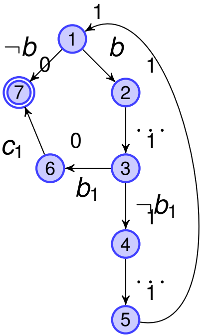

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- If no more registers free: Spill to memory
  - Expensive!
- Want to map as much variables as possible to registers

Example

```
1: x=M[a]
2: y=x+1
3: if (y=0) {
4:   z=x*x
5:   M[a]=z
   } else {
7:   t=-y*y
8:   M[a]=t
9: }
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- How many registers are needed?

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- How many registers are needed?
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- Variables: a, x, y, z, t.

Example

```
1: R1=M[R3]
2: R2=R1+1
3: if (R2=0) {
4:    R1=R1*R1
5:    M[R3]=R1
   } else {
7:    R1=-R2*R2
8:    M[R3]=R1
9: }
```

- How many registers are needed? Assuming all variables dead at 9
- Variables: a, x, y, z, t.
- Three registers suffice: $x, z, t \mapsto R_1, \quad y \mapsto R_2, \quad a \mapsto R_3$

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- Observation: Two variables can be mapped to same register, if their live ranges do not overlap

Example

```
                // L          a x y z t
1: x=M[a]       // {a}        1
2: y=x+1        // {a,x}      1 1
3: if (y=0) {   // {a,x,y}    1 1 1
4:   z=x*x      // {a,x}      1 1
5:   M[a]=z     // {a,z}      1     1
   } else {
7:   t=-y*y     // {a,y}      1   1
8:   M[a]=t     // {a,t}      1       1
9: }            // {}
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Interference graph

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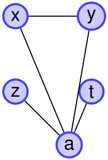

Last lecture

- Peephole optimization, removal of NOP-edges
- Linearization
  - Temperature of nodes = loop nesting depth
  - Preferably jump to colder nodes
- Register allocation
  - Minimal coloring of interference graph
    - NP-hard

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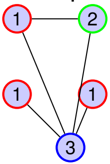

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  - Nodes of high degree first
  - Here: Pre-order DFS

Greedy heuristics, pseudocode

```
color(u):
  n = { v | (u,v) in E }
  c(u) = min i. i>=0 and forall v in n. i != c(v)
  for v in n
    if (c(v)==-1) color(v)

main:
  for u in V do c(u) = -1;

  for u in V do
    if c(u)==-1 then color(u)
```

Live Range Splitting

- Consider basic block,
  - i.e., sequence of statements, no jumps in/from in between
  - $(u, a_1, v_1), (v_1, a_2, v_2), \ldots, (v_{n-1}, a_n, v)$, with no other edges touching the v_i.

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- Example:

```
x=M[0] //
y=M[1] // x
t=x+y  // xy
M[2]=t //    t
x=M[4] //
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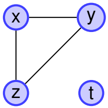
t=x+z  // x z
M[6]=t //    t
y=M[7] //
z=M[8] //  y
t=y+z  //  yz
M[9]=t //    t
```



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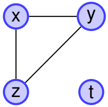

- Requires 3 registers

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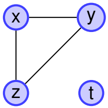

- Requires 3 registers
- But can do same program with two registers!

Live range splitting

$$
\begin{array}{ll}
x_1 = M[0] & // \\
y_1 = M[1] & //\ x_1 \\
t_1 = x_1 + y_1 & //\ x_1\ y_1 \\
M[2] = t_1 & //\ t_1 \\
x_2 = M[4] & // \\
z_1 = M[5] & //\ x_2 \\
t_2 = x_2 + z_1 & //\ x_2\ z_1 \\
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t_3 = y_2 + z_2 & //\ y_2\ z_2 \\
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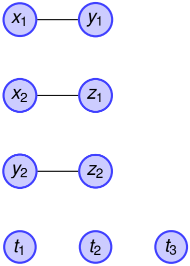

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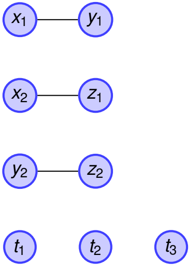

- In general: Rename variable if it is redefined

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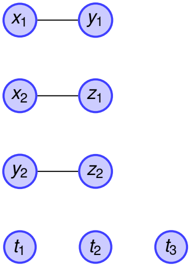

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- The interference graph forms an interval graph.

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    - Obviously, there is no coloring with less than $\max\{o(i) \mid i \in \mathbb{N}\}$ colors

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$\Longrightarrow$ Single static assignment form (SSA)

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  1. Transform program such that every program point v is reached by at most one definition of variable x which is live at v.
  2. Introduce a separate variant x_i for each definition of x, and replace occurrences of x by the reaching variants

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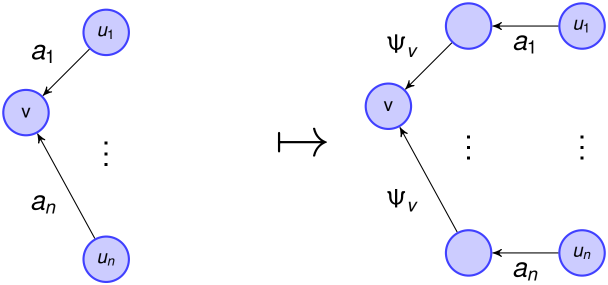

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  - Initial value: $R_0 := \{(x, v_0) \mid x \in \mathrm{Var}\}$
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  - Simultaneous assignment

Example

```
x:=M[I]
y:=1
while (x>0) {
  y=x*y
  x=x-1
}
M[R]=y
```

Example

```
1: x:=M[I]
2: y:=1
3: if not (x>0) goto 6;
4:   y=x*y
5:   x=x-1;
     goto 3
6: M[R]=y
7:
```

Example

```
1: x:=M[I]
2: y:=1
A: Nop    // Psi3
3: if not (x>0) goto 6
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B: Nop    // Psi3
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Example

```
1: x:=M[I]                  // {}    {(x,1),(y,1)}
2: y:=1                     // {x}   {(x,2),(y,1)}
A: Nop    // Psi3           // {x,y} {(x,2),(y,A)}
3: if not (x>0) goto 6;     // {x,y} {(x,2),(x,B),(y,A),(y,5)}
4:   y=x*y                  // {x,y} {(x,2),(x,B),(y,A),(y,5)}
5:   x=x-1                  // {x,y} {(x,2),(x,B),(y,5)}
B: Nop    // Psi3           // {x,y} {(x,B),(y,5)}
   goto 3
6: M[R]=y                   // {y}   {(x,2),(x,B),(y,A),(y,5)}
7:                          // {}    {(x,2),(x,B),(y,A),(y,5)}
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4:   y=x*y                  // {x,y}  {(x,2),(x,B),(y,A),(y,5)}
5:   x=x-1                  // {x,y}  {(x,2),(x,B),(y,5)}
B: x=x|y=y                  // {x,y}  {(x,B),(y,5)}
   goto 3
6: M[R]=y                   // {y}    {(x,2),(x,B),(y,A),(y,5)}
7:                          // {}     {(x,2),(x,B),(y,A),(y,5)}
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```
1: if (*) goto 3
2:    x=1
  goto 4
3:    x=2
4: if (*) goto 6
5:    M[0]=x
6: M[1]=x
7: HALT
```

Discussion

- This ensures that only one definition of a variable reaches each program point
  - Identifying the definitions by simultaneous assignments on edges to same join points
- However, we may introduce superfluous simultaneous definitions
- Consider, e.g.

```
1: if (*) goto 3
2:   x=1
A:   x=x
  goto 4
3:   x=2
B:   x=x
4: if (*) goto C
5:   M[0]=x
D:   x=x
6: M[1]=x
7: HALT

C: x=x
  goto 6
```

Improved Algorithm

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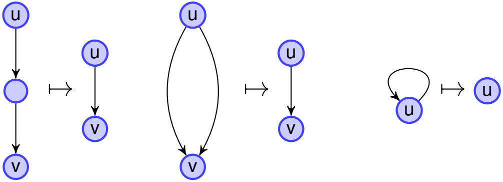
- The efficiency depends on the number of rounds
  - For well-structured CFGs, we only need one round
    - Example where 2 rounds are required on board.
  - We always may terminate after k rounds by using naive algorithm

Well-structured CFGs

- A CFG is well-structured, if it can be reduced to a single edge or vertex by the following transformations



Examples

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- Assume, each program point u is reached by exactly one definition $(x, w) \in R[u]$ for each variable x live at u
- Define $\Phi_u(x) := x_w$ for the w with $(x, w) \in R[u]$
- Transform edge (u, a, v) to $(u, T_{u,v}(a), v)$, where

$$
\begin{aligned}
T_{u,v}(\mathrm{Nop}) &= \mathrm{Nop} \\
T_{u,v}(\mathrm{Neg}(e)) &= \mathrm{Neg}(\Phi_u(e)) \\
T_{u,v}(\mathrm{Pos}(e)) &= \mathrm{Pos}(\Phi_u(e)) \\
T_{u,v}(x = e) &= x_v = \Phi_u(e) \\
T_{u,v}(x = M[e]) &= x_v = M[\Phi_u(e)] \\
T_{u,v}(M[e_1] = e_2) &= M[\Phi_u(e_1)] = \Phi_u(e_2) \\
T_{u,v}(\{x = x \mid x \in X\}) &= \{x_v = \Phi_u(x) \mid x \in X\}
\end{aligned}
$$

and $\Phi_u(e)$ applies Φ_u to every variable in e

Example

```
1: x:=M[0]
2: y:=1
A: x=x|y=y
3: if not (x>0) goto 6;
4:   y=x*y
5:   x=x-1
B: x=x|y=y
   goto 3
6: M[1]=y
7:
```

Example

```
1: x_2:=M[0]
2: y_A:=1
A: x_3=x_2|y_3=y_A
3: if not (x_3>0) goto 6;
4:   y_5=x_3*y_3
5:   x_B=x_3-1
B:   x_3=x_B|y_3=y_5
   goto 3
6: M[1]=y_3
7:
```

Register Allocation for SSA form

Theorem

Assume that every program point is reachable from start and the program is in SSA form without assignments to dead variables.
Let λ denote the maximal number of simultaneously live variables and G the interference graph of the program variables. Then:

$$\lambda = \omega(G) = \chi(G)$$

where $\omega(G), \chi(G)$ are the maximal size of a clique in G and the minimal number of colors for G, respectively.
A minimal coloring of G, i.e., an optimal register allocation can be found in polynomial time.

Background: Register allocation for SSA

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- Using a reverse perfect elimination ordering as node ordering for the greedy algorithm yields a minimal coloring
- For graphs in SSA form, the dominance relation induces a perfect elimination ordering on the interference graph
  - Thus, we do not even need to construct the interference graph:
  - Just traverse CFG with pre-order DFS, and assign registers first-come first serve.

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- And only perform optimizations that increase register pressure if still enough registers available

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- But what about more than two registers?

Discussion (ctd)

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  - Decompose into sequence of linear assignments and cyclic shifts

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  - Out-of-order execution: Processor may re-order instructions
- Or compiler checks independence (VLIW)

Exam

- You may bring in two handwritten A4 sheets
- We will not ask you to write OCaml programs

Last Lecture

- Register allocation
  - by coloring interference graph
  - by going to SSA-form
- Instruction level parallelism
  - Pipelining, superscalar architectures

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  1. Re-arrange independent instructions (in basic blocks)
  2. Increase size of basic blocks, to increase potential for parallelizing

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Data dependence graph

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  - As long as ordering respects dependence graph

Example

```
x=x+1
y=M[A]
t=z
z=M[A+x]
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```

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1: x=x+1
2: y=M[A]
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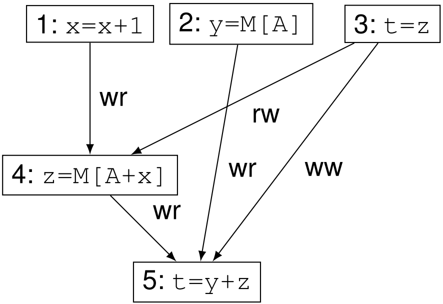

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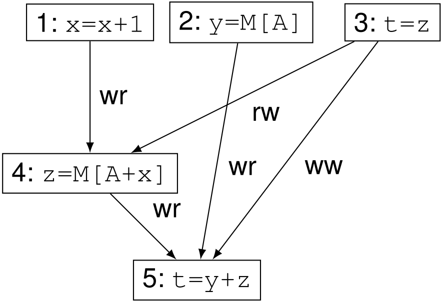

Possible re-ordering:

```
2: y=M[A]
1: x=x+1
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4: z=M[A+x]
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  - If node lies on critical path: Rise priority

Example: Live-range splitting

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1: x=r
2: y=x+1
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- Some processors do that dynamically

$\Longrightarrow$ Register renaming

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  i=i+1
  M[i] = 0
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}
if (i<n) {M[i]=0; i=i+1}  // For odd n
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short M [...];
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   (int)M[i] = 0
}
if (i<n) {M[i]=0; i=i+1}  // For odd n
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Discussion

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- Quick demo with gcc -O2 -funroll-loops

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    - E.g., different, statically allocated arrays
  - More sophisticated analyses, e.g., based on integer linear programming

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- Consider the following loop, assume A, B, C, D are guaranteed to be different

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for (i=0;i<n;++i) C[i] = A[i] + B[i];
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- Which may be further optimized to

```
for (i=0;i<n;++i) {
  R1 = A[i]; R2 = B[i];
  C[i] = R1 + R2;
  D[i] = R1 - R2}
```

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    - Iteration should iterate according to memory layout

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  - Possible implementation: Linked list

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  - This adds amortized constant extra cost
  - But makes cache-locality much better

Moving heap-allocated blocks to the stack

- Idea: Allocate block of memory on stack, instead of heap

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Moving heap-allocated blocks to the stack

- Idea: Allocate block of memory on stack, instead of heap
  - If pointers to this block cannot escape the current stack frame
  - Important for languages like Java, where almost everything is allocated on heap

Abstract example

```
int do_computation(...) {
  AuxData aux = new AuxData ()
  ...
  return ...
}
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- If no pointer to aux is returned or stored in global memory ...
- ... aux can be allocated on method's stack-frame

Example

- Recall our simple pointer-language. *Ret* is global variable.

```
1: x=new()
2: y=new()
  x[A] = y
  z=x[A]
  Ret = z
```

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2: y=new()
  x[A] = y
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```

- Allocation at 1 may not escape
- Thus we may do the allocation on the stack

In general

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  - Assigned to global variable
  - Reachable from global variable
- Forward may analysis. Same as pointer-analysis
  - Identify memory blocks with allocation sites
  - Analyze where variables/blocks may point to
  - If global variable/unknown memory block may point to block: Possible escape

Applying the optimization, heuristics

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  - e.g., not inside loop

Handling procedures more precisely

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  - Expensive
  - We do not always know whole program
    - E.g. Java loads classes at runtime
- In worst case: Assume everything visible to called procedure may escape
  - Which is consistent with parameter passing by global variables and previous analysis

Wrap-Up

- Several optimizations that exploit hardware utilization

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- A meaningful ordering

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     - Loop interchange, fission
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  2. Basic-block optimizations, to exploit instruction-level parallelism
     - Live-range splitting
     - Instruction scheduling
     - Loop unrolling, fusion
  3. Then register allocation
  4. And finally peephole optimization + instruction selection

Table of Contents

Last Lecture

- Optimizations to re-arrange memory access wrt. cache
  - Loop interchange
  - Lists vs. array-list
- Wrap-Up: Optimizations targeted towards features of hardware
- Started with functional languages

Functional language

- We consider simple functional language

```
prg ::= let rec f1 = e1 | ... | f_n = e_n in e
e ::= b | c | x | f_i | op | e e | fn x. e
    | let x=e in e
    | match e with p1 => e1 | ... | p_n => e_n
p ::= b | c x1 ... xn
```

Functional language

- We consider simple functional language

```
prg ::= let rec f₁ = e₁ | ... | f_n = e_n in e
e ::= b | c | x | f_i | op | e e | fn x. e
    | let x=e in e
    | match e with p₁ => e₁ | ... | p_n => e_n
p ::= b | c x₁ ... xn
```

- where
  - *b* is primitive constant
  - *c* is constructor
  - *x* is variable
  - f_i is recursive function
  - *op* is primitive operation

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Semantics

- Values $b, c\ v_1 \ldots v_n, \mathit{fn}\ x.\ e$ (Convention: v denotes values)

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- Values $b, c\ v_1 \dots v_n$, $fn\ x.\ e$ (Convention: v denotes values)
- Goal of semantics: Evaluate main expression to value
- Done by the following rules

$$[rec]\,\frac{\texttt{let rec f\_i = e\_i}}{f_i \to e_i}$$

$$[op]\,\frac{-}{op\ b_1 \dots b_n \to [\![op]\!](b_1, \dots, b_n)}$$

$$[app1]\,\frac{e_1 \to e_1'}{e_1\ e_2 \to e_1'\ e_2} \qquad [app2]\,\frac{e_2 \to e_2'}{v_1\ e_2 \to v_1\ e_2'}$$

$$[\beta{-}red]\,\frac{-}{(fn\ x.\ e)\ v \to e[x \mapsto v]}$$

$$[match1]\,\frac{e \to e'}{match\ e\ with\ \dots \to match\ e'\ with\ \dots}$$

$$[match2]\,\frac{-}{match\ v\ with\ \dots \to e_i\sigma} \quad (*)$$

$$[app{-}op]\,\frac{e_k \to e_k'}{op\ v_1\ \dots\ v_{k-1}\ e_k\ \dots e_n \to op\ v_1\ \dots\ v_{k-1}\ e_k'\ \dots e_n}$$

- where $let\ x = e_1\ in\ e_2$ is syntax for $(fn\ x.\ e_2)\ e_1$
- (*): $p_i{=}{>}e_i$ is the first pattern with $p_i\sigma = v$

Semantics

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  - Arguments are evaluated before function is called
- No types: Evaluation of badly-typed program just gets stuck
  - Example: `match 5 with True => ... | False => ...`

Example

```
let
  rec fac = fn x. match x with
    0 => 1
  | x => x * fac (x-1)
in fac 2

fac 2
```

Example

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let
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(fn x. ...) 2
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match 2 with ...
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- Note: Only simple patterns allowed in match

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Example (lazy)

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let
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and so on ...
```

Eager vs. Lazy

- Eager: Argument evaluated before function call
- Lazy: Function call before argument
    - Argument of match only until constructor is at top
        - Weak head normal form
    - Arguments of primitive operator: Completely

Optimization Plan

- Optimize on functional level

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- Translate to imperative language/IR

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- Optimize on functional level
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- Now: Optimizations on functional level

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Simple optimizations

- Idea: Move some evaluation from run-time to compile-time

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- Function-application to let

$\texttt{(fn x. e}_1\texttt{) e}_2\texttt{ --> let x=e}_2\texttt{ in e}_1$

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- Idea: Move some evaluation from run-time to compile-time
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  (fn x. e₁) e₂ --> let x=e₂ in e₁
  ```

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  ```
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- Let-reduction

  ```
  let x=e₁ in e --> e[x↦e₁]
  ```

Substitution

- Beware of name-capture

```
let x = 1 in
let f = fn y. x+y in
let x = 4 in
  f x
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- Convention: Substitution uses α-conversion to avoid name-capture

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  - Consider reduction of $f = \dots$.
- α-conversion: (Consistent) renaming of (bound) variables does not change meaning of program
- Convention: Substitution uses α-conversion to avoid name-capture
  - Here: Convert `let x=4 in f x` to `let x₁=4 in f x₁`

Termination issues

- Let-reduction may change semantics

```
let rec f = fn x. 1 + f x in
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- For eager evaluation, non-terminating programs may be transformed to terminating ones

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- For eager evaluation, non-terminating programs may be transformed to terminating ones
- For lazy evaluation, semantics is preserved

Side-effects

- Languages like SML/OCaml/F# have side-effects

Side-effects

- Languages like SML/OCaml/F# have side-effects
- Side-effecting expressions must not be let-reduced

```
let _ = print "Hello"
in ()
```

Application of let-reduction

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  - Re-computing values instead of storing them in variable
    ```
    let x=expensive-op in x+x
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    `let x=expensive-op in x+x`
- May blow up program code exponentially
  `let x = x+x in let x = x+x in ... in x`

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- May blow up program code exponentially
  `let x = x+x in let x = x+x in ... in x`
- Heuristics for application: reduce `let` x_1=e_1`in e`
  - if e_1 is a variable (or constant)

Application of let-reduction

- May make program less efficient
  - Re-computing values instead of storing them in variable
    `let x=expensive-op in x+x`
- May blow up program code exponentially
  `let x = x+x in let x = x+x in ... in x`
- Heuristics for application: reduce `let x₁=e₁in e`
  - if e_1 is a variable (or constant)
  - if x_1 does not occur in e

Application of let-reduction

- May make program less efficient
  - Re-computing values instead of storing them in variable
    `let x=expensive-op in x+x`
- May blow up program code exponentially
  `let x = x+x in let x = x+x in ... in x`
- Heuristics for application: reduce `let` x_1=e_1`in e`
  - if e_1 is a variable (or constant)
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  - if x_1 occurs exactly once in e

More transformations

- Valid for programs (fragments) with no side-effects

```
(let x=e in e1) e2 --> let x=e in e1 e2
  // Renaming x to avoid name capture

let x1=e1 in let x2=e2 in e
--> let x2=e2 in let x1=e1 in e
  // If x1 not free in e2
  // Renaming x2 to avoid name capture

let x1 = (let x2=e2 in e1) in e
--> let x2=e2 in let x1=e1 in e
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```

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- May open potential for other optimizations

Inlining

- Consider program `let f=fn x.` e_1 `in e`

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Example

```
let fmax = fn f. fn x. fn y.
  if x>y then f x else f y in
let max = fmax (fn x. x) in
  ...
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Example

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(inlined fmax)

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(Let-reduction for single-var expressions and unused variables)

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(Inlined w)

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((Safe) let-reduction (copy variables))

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let w = fn f. fn y. f (y f y) in
let fix = fn f. f (f (f (... f (w f w))))
```

(...)

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  - Only inlining terms of form `fn x. e`, which are not evaluated, unless applied to an argument
- In untyped languages (e.g., LISP), the inlining optimization may not terminate
- In typed languages like OCaml or Haskell, however, we have
  - Inlining always terminates

Table of Contents

Specialization of recursive functions

- Function to square all elements of a list
  - Note: Dropping the restriction that let-rec occurs outermost

```
let rec map = fn f. fn l.
  match l with
    [] => []
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let f = fn x. x*x in
let sqrl = map f in ...
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Function folding

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  - Potential for new specializations may be created
  - Infinitely often ...
  - `let rec f = fn g. fn l. ... f (fn x. g (g x)) ...`
- Safe and simple heuristics:
  - Only specialize functions of the form

    ```
    let rec f = fn x. e
    ```

    such that recursive occurrences of *f* in *e* have the form *f x*

Table of Contents

Deforestation

Deforestation

- Idea: Often, lists are used as intermediate data structures

Deforestation

- Idea: Often, lists are used as intermediate data structures
- Standard list functions

```
let rec map = fn f. fn l. match l with
  [] => []
| x#xs => f x # map f xs

let rec filter = fn P. fn l. match l with
  [] => []
| x#xs => if P x then x#filter P xs else filter P xs

let rec foldl = fn f. fn a. fn l. match l with
  [] => []
| x#xs => foldl f (f a x) xs
```

Deforestation

- Examples of derived functions

```
let sum = foldl (+) 0

let length = sum o map (fn x. 1)

let der = fn l.
  let n = length l in
  let mean = sum l / n in
  let s2 = (
      sum
    o map (fn x. x*x)
    o map (fn x. x-mean)) l
  in
    s2 / n
```

Idea

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- In general, we can define rules for combinations of the basic list functions like fold, map, filter, ...

```
map f o map g = map (f o g)
foldl f a o map g = foldl (fn a. f a o g) a
filter P o filter Q = filter (fn x. P x & Q x)
...
```

Idea

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map f o map g = map (f o g)
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filter P o filter Q = filter (fn x. P x & Q x)
...
```

- We may also need versions of these rules in first-order form, e.g. `map f (map g l) = ...`

Example

```
let der = fn l.
  let n = length l in
  let mean = sum l / n in
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      sum
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Let-optimization/ inlining

Example

```
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map-map rule

Example

```
let der = fn l.
  let n = length l in
  let mean = sum l / length l in
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      fn a. (+) a o (fn x. x*x) o (fn x. x-mean)
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fold-map rule

Example

```
let der = fn l.
  let n = length l in
  let mean = sum l / length l in
  let s2 = foldl (
      fn a. fn x. let x=x-mean in let x=x*x in a+x
    ) 0 l
  in
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```

function-application, unfolding of *o*, let-optimization.

Discussion

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- Need rules for many combinations of functions.
  - Does not scale
- Only works for built-in functions
  - Could try to automatically recognize user-defined functions
- Can be extended to algebraic datatypes in general
  - They all have standard map and fold functions

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  - But no list-reversal, as *foldl* can
- Problem: How to compose two foldr-calls?
  - `foldr f1 a1 (foldr f2 a2 l) = ???`

Composition of *foldr*

- Idea: Abstract over constructors

```
map f l = foldr (fn l. fn x. f x#l) [] l

map' f l = fn c. fn n.
  foldr (fn l. fn x. c (f x) l) n l
build g = g (#) []
map f l = build (map' f l)
```

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- Have

```
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```

  - If abstraction over list inside *g* done properly
  - I.e., *g* actually produces list using its arguments

Example

```
map f (map g l)
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= build (fn c. fn n.
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= build (fn c. fn n. map' g l (fn l. fn x. c (f x) l) n)
```

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- Applying a chain of functions: (*build foldr*) (*build foldr*) ... (*build foldr*)
- Can be re-bracketed to *build* (*foldr build*) ... (*foldr build*) *foldr*
- And the inner pairs cancel out, leaving a single *build foldr*

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  - Which is exactly the required abstraction over the list constructors

Wrap-up

- Transformations for functional programs

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  - Let-optimization

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  - Specialization
  - Deforestation
  - ...
- Aim at reducing complexity before translation to IR
- On (imperative) IR, all former optimizations of this lecture can be done
  - Important one: Tail-call optimization
    - There are no loops in functional languages

That's it!

Questions?