
WHENEVER I LEARN A NEW SKILL I CONCOCT ELABORATE FANTASY SCENARIOS WHERE IT LETS ME SAVE THE DAY.
OH NO! THE KILLER MUST HAVE FOLLOWED HER ON VACATION!
BUT TO FIND THEM WE'D HAVE TO SEARCH THROUGH 200 MB OF EMAILS LOOKING FOR SOMETHING FORMATTED LIKE AN ADDRESS!
IT'S HOPELESS!
EVERYBODY STAND BACK.
I KNOW REGULAR EXPRESSIONS.
PERL!
TAP TAP

Verified conversion between NFAs and regular expressions

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Problem

- **Given**
  - $r \in RE(\Sigma)$
  - $\mathcal{A} = (\mathcal{Q}, \Sigma, \Delta, \mathcal{I}, \mathcal{F})$
- **Wanted**
  - $\texttt{nfa\_to\_re} :: NFA(\mathcal{Q}, \Sigma) \Rightarrow RE(\Sigma)$
  - $\texttt{re\_to\_nfa} :: RE(\Sigma) \Rightarrow NFA(\mathcal{Q}, \Sigma)$
- **Such that**
  - $\mathcal{L}(\texttt{nfa\_to\_re}(\mathcal{A})) = \mathcal{L}(\mathcal{A})$
  - $\mathcal{L}(\texttt{re\_to\_nfa}(r)) = \mathcal{L}(r)$

Problem

Regex as defined in `Regular_set.thy`:

- *Zero* with $\mathcal{L}(\mathit{Zero}) = \{\}$
- *One* with $\mathcal{L}(\mathit{One}) = \{\varepsilon\}$
- *Atom a* with $\mathcal{L}(\mathit{Atom}\ a) = \{a\}$
- *Plus* r_1 r_2 with $\mathcal{L}(\mathit{Plus}\ r_1\ r_2) = \mathcal{L}(r_1) \cup \mathcal{L}(r_2)$
- *Times* r_1 r_2 with $\mathcal{L}(\mathit{Times}\ r_1\ r_2) = \mathcal{L}(r_1) \cdot \mathcal{L}(r_2)$
- *Star* r with $\mathcal{L}(\mathit{Star}\ r) = \mathcal{L}(r)^*$

Algorithm 1 – system of equations

- Set up system of equations for accepted words in state i:

  $X_i = \{c_1\} \cdot X_{i_1} \cup \ldots \cup \{c_k\} \cdot X_{i_k} \quad (\cup \{\varepsilon\})$
- While $\exists$ unsolved equation for some X_i:
  1. bring equation into form $X_i = A \cdot X_i \cup B$
  2. apply Arden's lemma: $X_i = A^* \cdot B$
  3. propagate solution to other equations

Algorithm 2 – recursive approach

Definition

Let $R^{i,j}_{Q'} \in RE(\Sigma)$
such that $\mathcal{L}(R^{i,j}_{Q'}) = \mathcal{L}_{(Q',\Sigma,\delta,I,F)}(i,j)$

Then: recursive computation of $R^{i,j}_{Q}$

- $R^{i,i}_{\{\}} = \delta(i,i) \mid \varepsilon$
- $R^{i,j}_{\{\}} = \delta(i,j) \quad (\text{for } i \neq j)$
- $R^{i,k}_{Q' \cup \{j\}} = R^{i,k}_{Q'} \mid R_{Q'}(i,j) \cdot R_{Q'}(j,j)^* \cdot R_{Q'}(j,k)$

Algorithm 3 – matrix method (cf. Dexter Kozen)

System can be seen as $AX = b$

where $X = (X_1, \ldots, X_n)^T$ and $A \in RE(\Sigma)^{n \times n}$

Solution: $X = A^* \cdot b$

Compute A^* by Divide & Conquer

Note: all previous algorithms can be seen as special case of this Divide & Conquer algorithm

Algorithm 4 – contractions

Definition (GNFA (Generalised NFA))

Properties of a GNFA:

- *regexes as labels,* $\delta : Q \mapsto Q \mapsto RE(\Sigma)$
- $I = \{Start\}$ *and* $F = \{End\}$
- $\forall q.\ \delta(q, Start) = \{\} \ \wedge \ \delta(End, q) = \{\}$

Idea:

1. convert NFA to GNFA (trivial)
2. remove some state q and compensate with new transitions
3. rinse, repeat

Algorithm 4 – contractions

We have implemented contractions. Why?

- simple, intuitive algorithm
- straightforward correctness proof
- formalisation effort significant, but not higher than the other algorithms
- implementation equivalent to all the other algorithms (except matrix)
- complexity $O(n^3)$ the same for all algorithms
  But: worst-case output size $2^{\Omega(n)}$ (*Gruber/Holzer*, 2008)

Abstract algorithm

Note: on the abstract level, GNFA is labelled with elements from Σ^* not $RE(\Sigma)$.

Why? all proofs far easier; $\cup$ is commutative and associative, *Plus* isn't. Regexes have too much "structure" for the abstract level.

Abstract algorithm

$\mathcal{A}' \leftarrow \mathit{NFA\_to\_GNFA}\ \mathcal{A}$
while $Q_{\mathcal{A}'} \neq \{\mathit{Start}, \mathit{End}\}$ **do**
  obtain $q \in Q_{\mathcal{A}'} \setminus \{\mathit{Start}, \mathit{End}\}$
  for all $u \in \delta(u, q) \neq \{\}$ **do**
    for all $v \in \delta(q, v) \neq \{\}$ **do**
      $\delta(u, v) := \delta(u, v) \ \cup \ \delta(u, q) \cdot \delta(q, q)^* \cdot \delta(q, v)$
    end for
  end for
  $Q := Q \setminus \{q\}$
end while
return $\delta(\mathit{Start}, \mathit{End})$

Abstract correctness proof

Abstract part of the theory:

1. definition of GNFA
2. inductive for reachability in GNFAs
3. lemma: adding the *subsumed transitions* of state q does not change behaviour of automaton
4. lemma: removing q thereafter does not change behaviour (reachability) of remaining automaton
5. definition of conversion abstract NFA $\rightarrow$ abstract GNFA

Refinement

1. implementation of algorithm as shown above
   - while loop invariant: $\forall u, v \in Q_{\mathcal{A}'}.\ \mathcal{L}_{\mathcal{A}'}(u, v) = \mathcal{L}_{\mathcal{A}}(u, v)$
   - after the loop, $Q = \{Start, End\}$ and therefore $\mathcal{L}_{\mathcal{A}'} = \delta(Start, End)$
2. Refinement: caching of predecessors/successors
3. Refinement: δ from $Q \mapsto Q \mapsto \Sigma^*$ to $Q \mapsto (Q \mapsto RE(\Sigma)\ option)\ option$

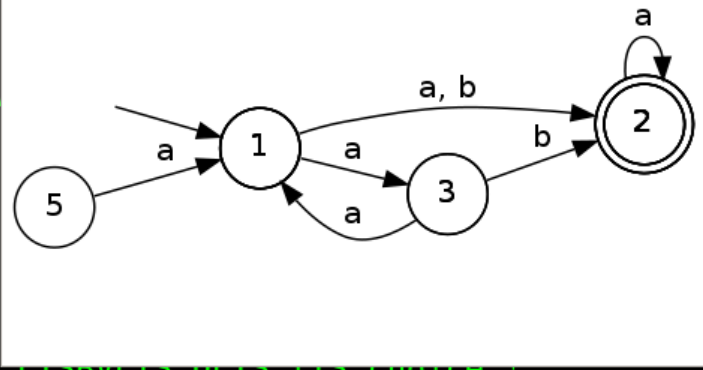
   Note: now the return value is a regex
4. Refinement: executable data structures

Evaluation

Exported ML code + sugar:

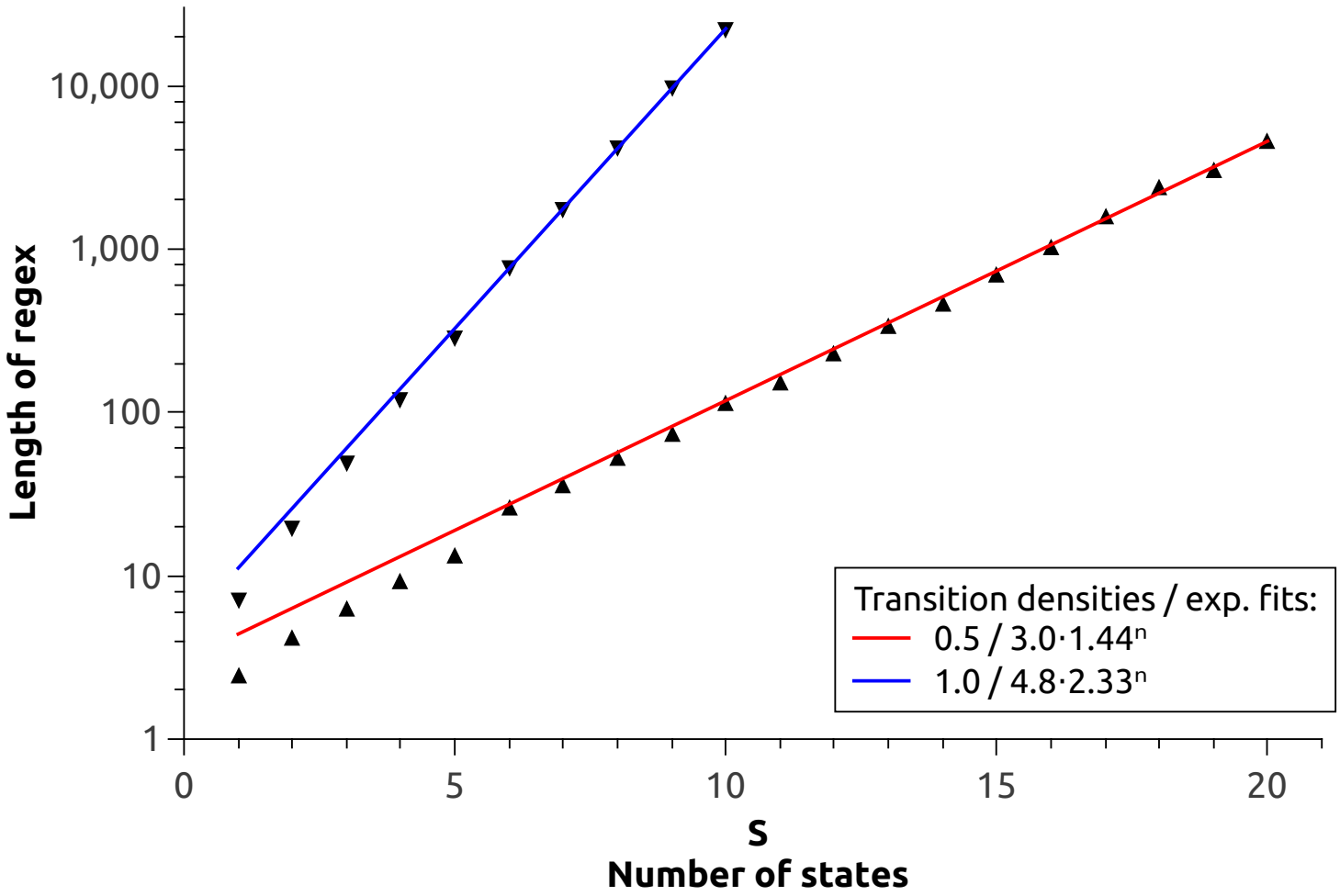
```
manuel@glados: ~/isaworspace/cava/Regexp/NFAtoRE/c
File  Edit  View  Search  Terminal  Help
        (Rbt (Branch (B, Empty, 2, (), Empty
         Nfa_props_ext (false, false, ())))))
> show_NFA automaton1;
Error-Type error in function application.
   Function: show_NFA : string option -> NF
   Argument: automaton1 : NFA
   Reason:
      Can't unify string option to
      (int, unit) RBT.rbt *
      ((string, unit) RBT.rbt *
      (((int, (string, (int, unit) RBT.rbt)
      (int, (string, int) RBT.rbt) RBT.rbt)
      ((int, unit) RBT.rbt *
      ((int, unit) RBT.rbt * unit NFAByLTS.nfa_props_ext))))
      (Incompatible types)
Found near show_NFA automaton1
Static Errors
> show_NFA NONE automaton1;
val it = (): unit
> rexp_to_string (nfa_to_rexp automaton1);
val it = "(aa)*(a|b|ab)a*": string
>
```

graphviz:



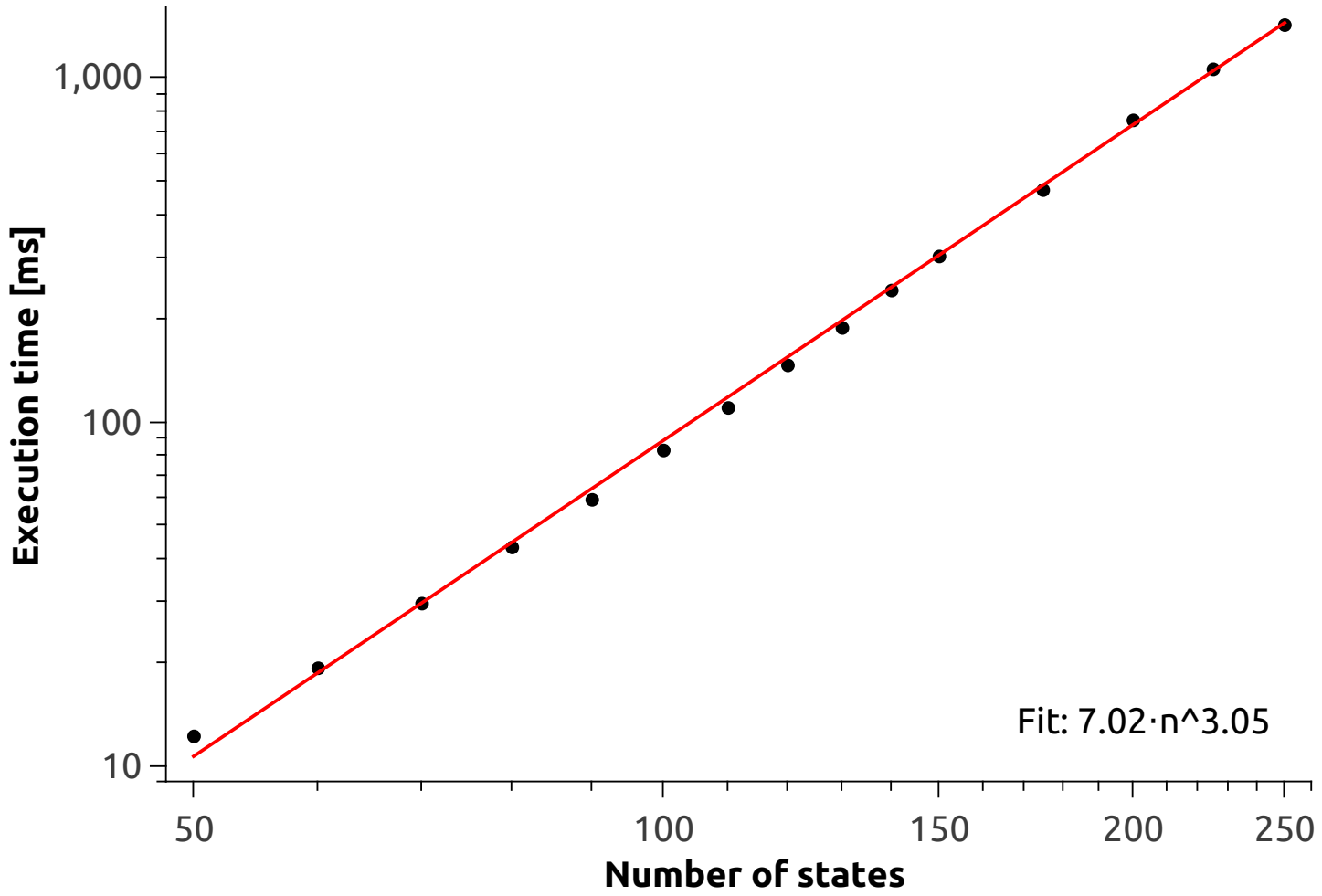
Evaluation

Average size of regex against # of states, log plot



Evaluation

Average computation time against # of states, log/log plot



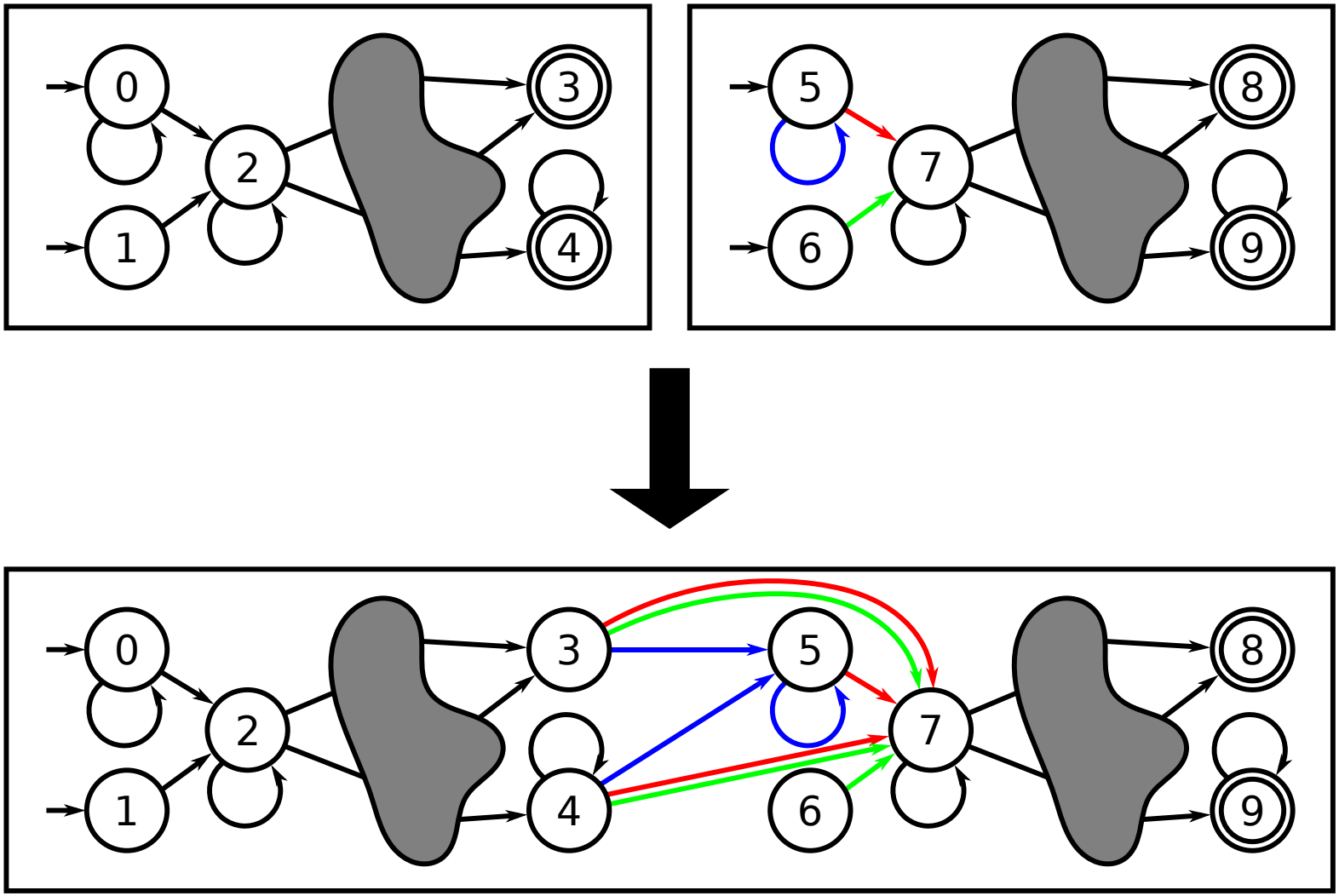
Classic Thompson Algorithm

- Follows the syntactic structure of the regular expression
- Constructs automata for primitives *Zero*, *One* and *Atom a*
- Given a non-primitive expression
  - Constructs automata for the subexpressions using recursion
  - Merges these automata according to the original expression
- Usually described in terms of ε-NFAs

Algorithms Based on Derivatives

- Derivative of r with respect to u:
  $\mathcal{L}(D_u(r)) = \{v | uv \in \mathcal{L}(r)\}$
- $D_u(r)$ is a regular expression
- Algorithms
  - Brzozowski
  - Glushkov / McNaughton & Yamada
  - Berry & Sethi

Thompson Merging Algorithm



Thompson Merging Realization

- Define algorithm in terms of sets and abstract NFAs
  - Primitives: `TM_zero_NFA` `TM_one_NFA` `TM_plus_NFA`
  - Composite: `TM_plus_NFA` `TM_times_NFA` `TM_star_NFA`
  - Translation: `TM_translate_NFA`
  - States are natural numbers
  - Automata are built using consecutive state indices
- Prove correctness of abstract definitions
  - Example: $\mathcal{L}(\texttt{TM\_star\_NFA}\,(r)) = \mathcal{L}(r)^*$
- Refine to executable data structures

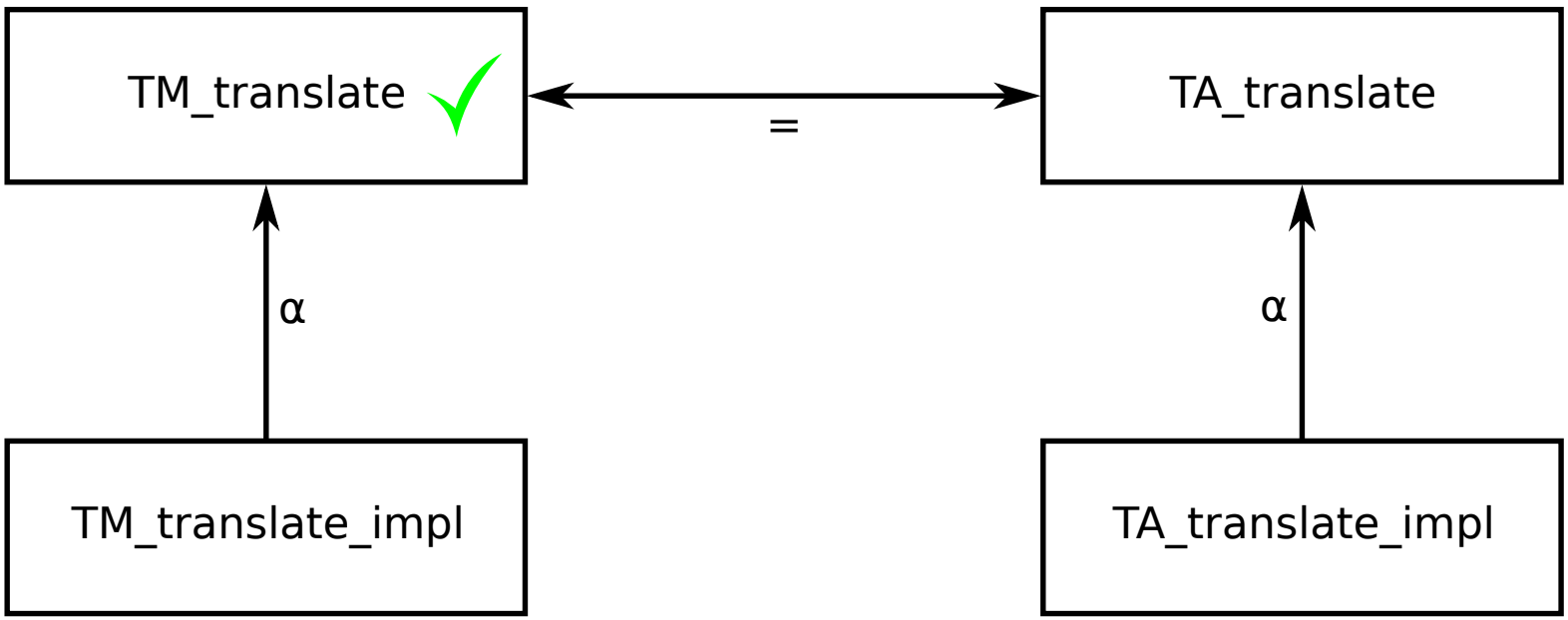
Thompson Accumulating Algorithm

- Union of large sets performs poorly
- Idea: Collect small automata in accumulator automaton
- TM: Merge given automata, return result
- TA: Build merged automaton on top of accumulator, return new accumulator
- Further optimizations: Accumulator state
  - gives the first unused state index
  - indicates whether the accumulator automaton accepts ε

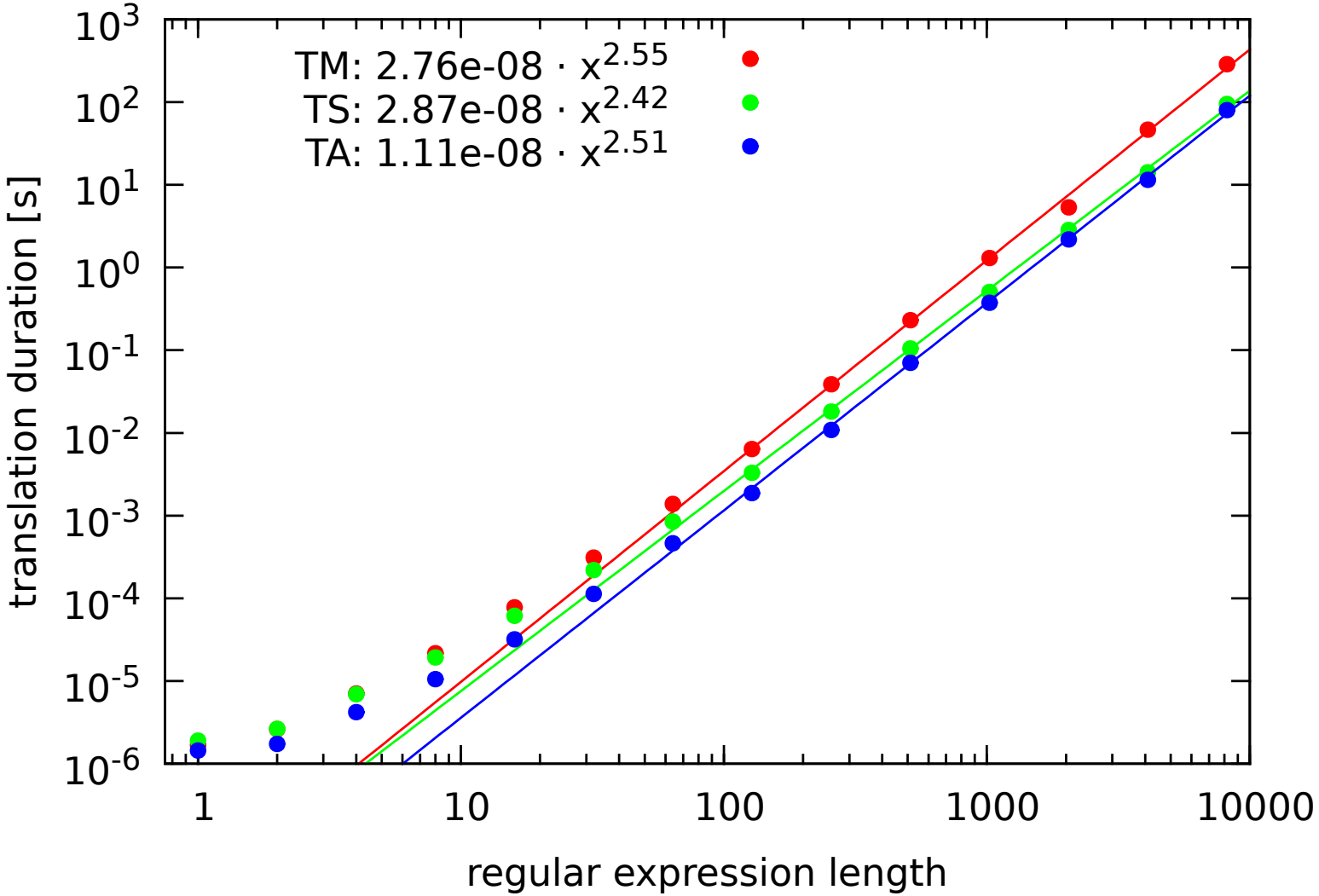
Thompson Accumulating Realization

- Define algorithm in terms of sets and abstract NFAs
- Prove that the TA algorithm builds the same automaton as the TM algorithm
- Refine to executable data structures

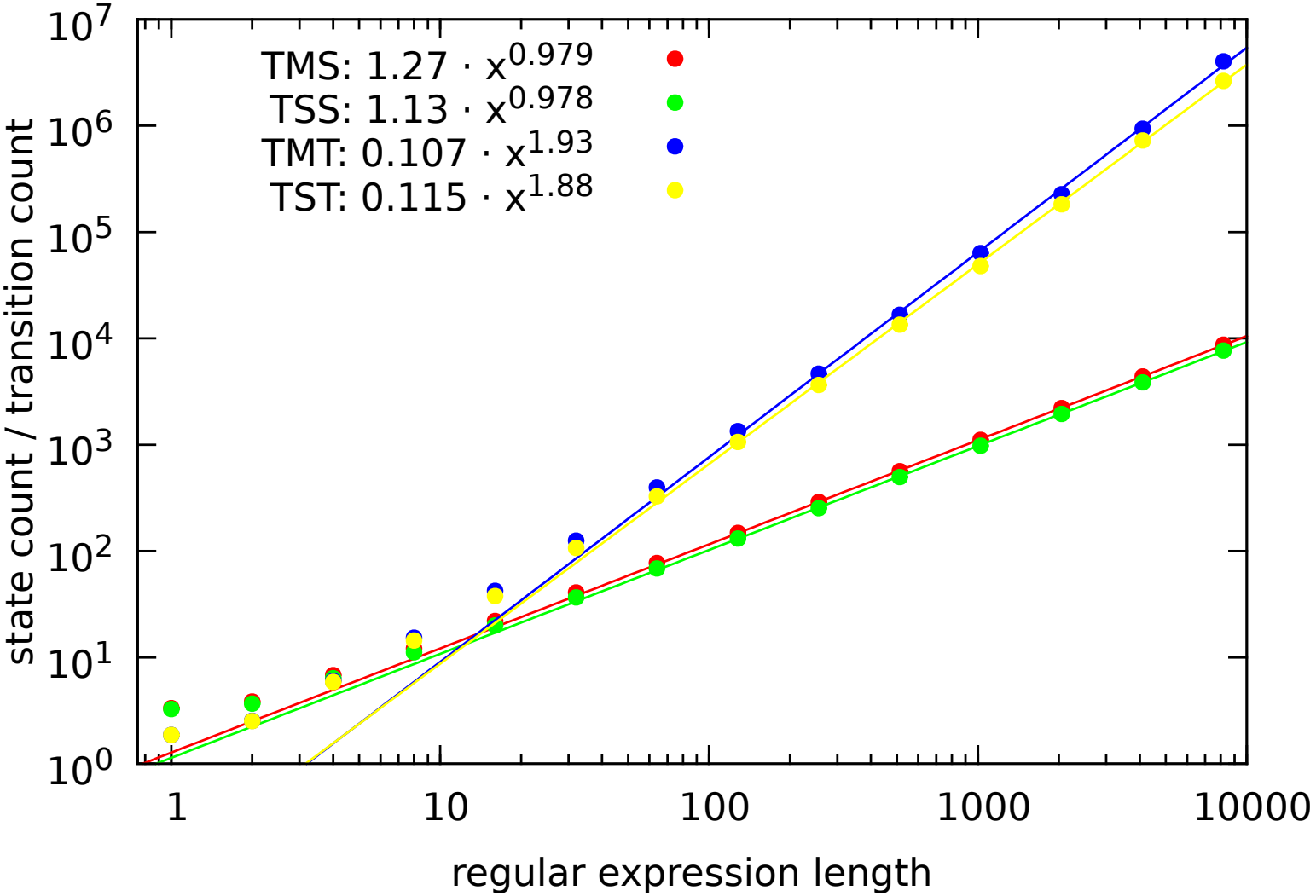
Algorithms and Correctness Proofs



Translation Performance



Automaton Size



Conclusion

Possible improvements / further work:

- General
  - clean up proofs, make an AFP entry
  - what about extended regexes ($\cap$, $\neg$)?
- NFA to RE
  - use on-the-fly simplification of regexes to avoid unnecessarily bloated regexes for complex automata
- RE to NFA
  - implement a derivative-based algorithm if smaller automata and faster translation times are desired