Parenthesis-property of DFS-discovery and -finishing times (Thm 23.6 of CLR):
For any two nodes u,v of a directed graph, if d[u] < d[v] (i.e. u discovered before v), either f[v] < f[u] (i.e. visit-time-interval for v subset of interval for u) and then v is descendant of u in DFS-tree, or f[u] < d[v] (i.e. visit of u finished before the search of v begins).

Proof: if not f[u] < d[v], then u still grey while v is being visited. DFS-visit(v) will then terminate before DFS-visit(u).

Note: Each edge (u,v) of the graph will be visited once during DFS.

Classification of edges: according to colour of endpoint during DFS-visit:

- *white* edge (u,v): white endpoint v (also called *tree* edge, with u=p[v])
- *grey* edge (u,v): grey endpoint v (also called *back* edge)
- *black* edge (u,v): black (already finished) endpoint v (sometimes called *forward/cross* edge).

Thm 23.7 of CLR (White-Path Theorem):
In a depth-first forest of a (directed or undirected) graph G=(V,E), vertex v is a descendant of vertex u in the DFS tree if and only if at the time d[u] that the search discovers u, vertex v can be reached from u along a path consisting entirely of white vertices.

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Proof: If v is a descendant of u, let w be any vertex on the path between u and v in the depth-first tree, so that w is a descendant of u. By the parenthesis theorem, $d[u] < d[w]$, so w is white at time d[u].
Conversely, suppose that v is reachable from u along a path of white vertices at time d[u], but v does not become a descendant of u in the depth-first tree. Without loss of generality, assume that every other vertex along the path becomes a descendant of u. (Otherwise, let v be the closest vertex to u along the path that does not become a descendant of u.) Let w be the predecessor of v in the path, so that w is a descendant of u (w and u may in fact be the same vertex), and by the parenthesis theorem $f[w] \leq f[u]$ (where $f[w]=f[u]$ if $u=v$). Note

that v must be discovered after u is discovered (because v is white at time d[u]), but before w is finished (because when DFS-visiting the neighbours of w, which include v, either v has already been discovered or v is white and will then be discovered, and w will finish afterwards). Therefore, $d[u] < d[v] < f[w] \leq f[u]$. Then the interval [d[v], f[v]] is contained

entirely within the interval [d[u], f[u]] so that v is after all a descendant of u.

Thm 23.9 of CLR: An *undirected* graph G has no black edges (i.e. only white or grey edges, i.e. tree or back edges)

Proof: (u,v) edge of G, w.l.o.g. $d[u] < d[v]$. Then $f[u] < d[v]$ not possible since v is a neighbour of u, so DFS-visit of u cannot have finished when v is discovered.
Hence $f[v] < f[u]$ by Parenthesis theorem. If the edge is explored in DFS as (u,v), it becomes a white (= tree) edge, if as (v,u), a grey (= back) edge.

Lemma 23.10 of CLR: A directed graph G has *no cycle* if and only if a DFS of G gives no grey (= back) edges.

Proof: A grey edge clearly gives a cycle. Conversely, consider a cycle and let v be the node with smallest d[v] in that cycle (the node that is discovered first).
Let u be its predecessor in the cycle, so that (u,v) is an edge.
Claim: this edge is grey, i.e. while exploring u the node v is still grey.
Reason: at time d[v], there is a path of entirely white vertices (along the cycle) to u, so by white-path theorem, u becomes descendant of v in DFS and (u,v) is a back edge.

Topological-Sort(G) for digraph G:

1. Perform DFS(G), where
2. Whenever a node is finished (marked black), insert it at the *beginning* of a list.
3. Output the list, which is then topologically sorted (in effect in reverse order of their finishing times).

Running time $O(|V|+|E|)$.

Why does topological sort work? Consider an edge (u,v) of G when encountered during DFS. Then v cannot be grey by Lemma 23.10 since G is acyclic, so v is either black, i.e. already finished and $f[v] < f[u]$, or white, which means that (u,v) is a tree edge and v hence a DFS-descendant of u, and thus again $f[v] < f[u]$ by the parenthesis theorem.