

The Isabelle Framework

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1 Overview

Isabelle, which is available from <http://isabelle.in.tum.de>, is a generic framework for interactive theorem proving. The *Isabelle/Pure* meta-logic allows the formalization of the syntax and inference rules of a broad range of object-logics following the general idea of natural deduction [32, 33]. The logical core is implemented according to the well-known “LCF approach” of secure inferences as abstract datatype constructors in ML [16]; explicit proof terms are also available [8]. *Isabelle/Isar* provides sophisticated extra-logical infrastructure supporting structured proofs and specifications, including concepts for modular theory development. *Isabelle/HOL* is a large application within the generic framework, with plenty of logic-specific add-on tools and a large theory library. Other notable object-logics are *Isabelle/ZF* (Zermelo-Fraenkel set-theory, see [34, 36]) and *Isabelle/HOLCF* [26] (Scott’s domain theory within HOL). Users can build further formal-methods tools on top, e.g. see [53].

Beginners are advised to start working with *Isabelle/HOL*; see the tutorial volume [30], and the companion tutorial [28] covering structured proofs. A general impression of *Isabelle/HOL* and *ZF* compared to other systems like Coq, PVS, Mizar etc. is given in [52]. The Proof General Emacs interface [3] is still the de-facto standard for interaction with Isabelle. The Isabelle document preparation system enables one to generate high-quality PDF- \LaTeX documents from the original theory sources, with full checking of the formal content.

The *Archive of Formal Proofs* <http://afp.sf.net> collects proof libraries, examples, and larger scientific developments, mechanically checked with Isabelle. AFP is organized like a journal everybody can contribute to. Submitting formal theories there helps to maintain applications in the longer term, synchronized with the ongoing development of Isabelle itself.

2 Specification Mechanisms

Isabelle/Pure is a minimal version of higher-order logic; object-logics are specified by stating their characteristic rules as new axioms. Any later additions in application theories are usually restricted to *definitional specifications*, and the desired properties are being proven explicitly. Working directly from primitive definitions can be tedious, and higher-level specification mechanisms have emerged over the years, implemented as derived concepts within the existing background logic. This includes (co)inductive sets [35], inductive datatypes [11], and recursive functions [42, 23].

3 Structured Proofs

The Isar proof language [49] continues the natural deduction principles of Isabelle/Pure, working with human readable proof texts instead of primitive inferences. “*Isar*” abbreviates “*Intelligible semi-automated reasoning*”; the language is also related to Mizar, but many underlying principles are quite different [54].

The Isabelle/Isar design also follows the generic framework idea [51]. Starting with a small selection of common principles of natural deduction, various advanced concepts are defined as derived elements (e.g. for calculational reasoning [7] and complex induction proofs [50]). The demands for structured proof composition have also influenced the way of writing definitions and statements, using extra language elements corresponding to Isar proofs, instead of going through the object-logic again [12].

4 Modular Theory Development

Isabelle theories are organized as a graph, with monotonic operations to extend and to merge logical environments. Smaller units of context elements are managed by separate mechanisms for modular theory development, notably *axiomatic type-classes* [27, 48, 40] and *locales* [21, 5, 6]. More recent work integrates type-classes and locales [18], joining the simplicity of classes with the flexibility of locales.

The generic *local theory* concept [19] integrates user-defined module mechanisms smoothly into the Isabelle/Isar framework. The Isabelle distribution already incorporates locales, classes, and class instantiation contexts into the local theory infrastructure. Other approaches to modular theories like AWE [13] could be integrated as well.

Internally, all aspects of locality in Isabelle are centered around the notions of *proof context* and *morphism* — to transfer entities from one context into another. This covers primitive types / terms / theorems of Isabelle/Pure, and any extra-logical context data defined in Isabelle/Isar. This idea of “local everything” allows us to implement tools within an abstract theory and apply them in concrete application contexts later on. One example is an implementation [15] of algebraic methods on abstract rings that can be used for concrete rings.

5 Reasoning Tools

Isabelle has traditionally supported a fair amount of automated reasoning tools. The basic framework is centered around higher-order unification. The Simplifier supports higher-order rewriting, with plug-in interfaces for extra simplification procedures written in ML. The Classical Reasoner [37] and Classical Tableau Prover [38] have been recently complemented by the Metis prover due to Joe Hurd. Various arithmetic proof procedures are available as well. Sledgehammer [41] uses external automated provers (E, Vampire, SPASS) as untrusted search tools to find the necessary lemmas for a particular goal; the actual proof is then performed internally with Metis.

6 Counterexample Search

Because much of the time one (unwittingly) tries to prove non-theorems, Isabelle/HOL offers two facilities to find counterexamples: *Quickcheck* [10] tries randomized instantiation of the free variables and is restricted to executable formulae (see §7). *Refute* [47] searches for small finite countermodels by translating (unrestricted) HOL formulae into propositional logic and hitting them with a SAT solver.

7 Code Generation

Executable HOL theories, including recursive functions and inductive definitions, can be translated into various functional languages, notably SML, OCaml, Haskell [9, 17]. Efficient imperative code can be generated from functions written in monadic style [14]. Results of ML-level computations can be re-imported as theorems (“reflection”) to allow efficient computations in proofs. These code generators are restricted to evaluation of closed terms. Efficient evaluation of terms with free variables is supported by a compiled implementation of “normalization by evaluation” [1].

8 Major Applications

In the past 20 years, Isabelle has been used by numerous researchers and students of computer-science and mathematics world wide. Below we summarize some representative large-scale applications.

Pure Mathematics. Here the largest applications are: a) The verification by Bauer, Nipkow [29] and Obua [31] of two of the algorithmic parts of Hales’ proof of the *Kepler Conjecture* (What is the densest arrangement of spheres in space?). This is part of Hales’ *Flyspeck* project, the complete verification of his proof. b) Avigad’s verification of the *Prime Number Theorem* [4] (about the distribution of primes). c) Paulson’s proof [39] of the relative consistency of the axiom of choice in ZF, formalized in Isabelle/ZF.

Systems verification. The *Verisoft* project <http://www.verisoft.de> formalized a whole computer system from the hardware up to an operating system kernel [2] and a compiler for C-dialect [24]. The *L4.verified* project [20, 43] verifies the L4 operating system microkernel, relating an abstract specification, a Haskell model, and the C code.

Programming languages. A large amount of work has gone into formalizations of a sequential Java-like language *Jinja* [22], including bytecode-verification, virtual machine and compiler. Jinja has become the basis for further extensions like multithreading [25] and multiple inheritance [46]. *Isabelle/HOL-Nominal* [44] extends Isabelle/HOL with a unique infrastructure for defining and reasoning about languages with bound variables. Many case studies have been carried out, for example about the meta theory of LF [45].

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